Abstract
For many applications, e.g. the daily tracking of market prices in front office systems, it is sufficient to consider a simplified CDO pricing model. In Volume I of this series such a simplified model was presented based on the assumption of a very large, homogeneous portfolio. However, e.g. for more detailed scenario analyses and the computation of hedge ratios, one has to relax these simplifying assumptions. In the current paper, several extensions of the CDO pricing model based on a Gumbel copula are discussed and possible implementations are presented. These extensions can be used to price CDO contracts on non-homogeneous pools.

1 Introduction
In Volume I of this series, we presented a model for the pricing of CDO tranches based on a latent market factor with heavy-tailed \( \alpha \)-stable distribution. The pricing routines developed in that paper were built on several simplifying assumptions rendering the numerical implementation efficient. However, for several further applications and investigations an extension of the model to non-homogeneous portfolios is needed. Possible applications include, e.g., detailed scenario analyses or hedge ratio computations. The purpose of this article is to describe possible ways to relax some of the simplifying assumptions and to investigate the resulting changes for model prices. Therefore, the paper is more concerned with numerical algorithms and a bit technical.

The large homogeneous portfolio (LHP) approximation relies on the following battery of simplifying assumptions. All notations are the same as in Volume I, which we presuppose as known by the reader.

(i) The portfolio is very large and all portfolio weights are the same, i.e. \( d \gg 2 \) and \( \omega_1 = \ldots = \omega_d = 1/d \).

(ii) All recovery rates are identical and deterministic, i.e. \( R_1 = \ldots = R_d =: R \in [0, 1] \).

(iii) All default times \( X_k \) of the portfolio constituents have the same marginal distribution function, which we denote by \( p(t) \), i.e. \( F_1(t) = \ldots = F_d(t) =: p(t) \) for all \( t \geq 0 \).

(iv) The components of \( (X_1, \ldots, X_d) \) are conditionally independent given a latent market factor.

We will present several ways to relax some of the above assumptions:
(a) In Section 2, we will relax the assumption of a very large portfolio and (iii).

(b) In Section 3, we will sketch how to additionally relax the assumptions of equal portfolio weights and identical and deterministic recovery rates. Furthermore, we will briefly explain how one could include a non-homogeneous dependence structure, whose implementation, however, then would have to rely on time-consuming Monte Carlo techniques.

We start with a short introduction of the model considered in the following. The current paper is based on the Gumbel copula model, which is equivalent to the model considered in Vol. I, in the sense as described in Remark 3.1 of Vol. I, but considerably easier to extend to the non-homogeneous case. Dependence is introduced by a common factor $M \sim S(1, \alpha)$. The vector of default times $(X_1, \ldots, X_d)$ is formally defined by

$$X_k := \inf \{ t > 0 : M h_k(t) > \epsilon_k \}, \quad k = 1, \ldots, d,$$

where $\epsilon_1, \ldots, \epsilon_d$ are iid unit exponential random variables and $h_k(t) := (-\log(1 - F_k(t)))^{1/\alpha}$. Consequently, the components of $(X_1, \ldots, X_d)$ are conditionally independent given $M$ and have the Gumbel copula as survival copula.

2 Inhomogeneous marginals

We now consider the following setting: The portfolio size is fixed, e.g., $d = 125$, and the marginal distribution functions, respectively default probabilities, are not necessary identical, given by $p_i(t), 1 \leq i \leq d$. All other simplifying assumptions still hold. This extension is quite natural whenever CDS contracts on the portfolio constituents are available, which then allow to draw conclusions about the corresponding marginal default probabilities.

Our scenario is the following: We want to price several (here five for simplicity) tranches of a CDO at once with attachment points $(l_1, u_1), \ldots, (l_5, u_5)$ and relevant payment dates $T_1, \ldots, T_k$. As explained in Vol. I of this series, required are the values in the following matrix:

$$TE := \begin{pmatrix}
TE_{l_1, u_1}(T_1) & \cdots & TE_{l_5, u_5}(T_1) \\
\vdots & \ddots & \vdots \\
TE_{l_1, u_1}(T_k) & \cdots & TE_{l_5, u_5}(T_k)
\end{pmatrix}.$$ 

One could also compute the portfolio loss distribution for every $T_i, 1 \leq i \leq k$, and compute the above matrix from those distributions. Here, however, we want to compute the different relevant tranche expectations directly. As usual, this is done by considering everything conditioned on the common factor and integrate out. The common factor considered here is actually only one $\alpha$-stable random variable $M$. We have

$$TE_{l_i, u_i}(T_j) \mid \{M = m\} = \mathbb{E} \left[ \min \{ u_i - l_i, \max \{ 0, L_{T_j}^{(d)} - l_i \} \} \mid M = m \right]$$

$$= \sum_{n=1}^d \min \{ u_i - l_i, \max \{ 0, \frac{1 - R}{d} n - l_i \} \} \mathbb{P}(L_{T_j} = n \mid M = m),$$
where $\tilde{L}_{T_j} := \sum_{k=1}^d 1\{X_k \leq T_j\}$, i.e. the number of companies defaulted until $T_j$. Conditioned on $M$, the $1\{X_k \leq T_j\}$ are independent with default probabilities

$$p_k(T_j|M = m) = 1 - \exp \left( -m \left( -\log(1 - F_k(T_j)) \right)^{1/\alpha} \right).$$

Thus, it is theoretically quite easy to compute the distribution of $\tilde{L}_{T_j}$ conditioned on $M = m$, and from that the value of the tranche expectations for $T_j$ conditioned on $M = m$. In practice, to compute the distribution of $\tilde{L}_{T_j}$ conditioned on $M = m$ efficiently, we will use the first approach presented in Hull, White (2004).

For a detailed explanation of the algorithm used to compute the distribution of a sum of independent Bernoulli random variables, see Hull, White (2004). The main idea is to use the representation

$$P(\tilde{L}_{T_j} = n|M = m) = P(\tilde{L}_{T_j} = 0|M = m) U(n),$$

where

$$U(n) := \sum_{I \subset \{1, \ldots, d\}: |I| = n} \prod_{r \in I} w_r,$$

with

$$w_r = \frac{p_r(T_j|M = m)}{1 - p_r(T_j|M = m)}.$$

This representation can be easily derived starting from

$$P(\tilde{L}_{T_j} = n|M = m) = \sum_{I \subset \{1, \ldots, d\}: |I| = n} \left( \prod_{r \in I} p_r(T_j|M = m) \prod_{r \in I^c} (1 - p_r(T_j|M = m)) \right),$$

factoring out the required quantities. For an efficient computation of $U$, Hull, White (2004) use an algorithm based on the so-called Newton-Girard formulas. Having a closer look at their algorithm, it becomes clear that it runs into numerical instabilities as soon as $w_r > 1$ for at least one $r$. A solution to this problem is found by dividing the portfolio into two subportfolios, one for which all $w_r$’s are smaller than one, and the rest. For the first portfolio, we can directly apply the algorithm of Hull, White (2004). For the second, we apply the analogous algorithm to derive the distribution of the number of surviving companies, instead of the number of defaulted companies. This switches the $w_r$ to $1/w_r$, making the algorithm stable again. In a last step, one has to “merge” the distributions of the two independent subportfolios.

Based on the previous procedure, we are able to compute $TE|\{M = m\}$. In a last step, we have to compute the vector-valued integral

$$TE = \int_0^\infty TE|\{M = m\} f_M(m) \, dm$$

by numerical quadrature. To compute the required density $f_M$ of the $\alpha$-stable random variable $M$, we implement the formula developed in Bernhart et al. (2013). Of course, one has to truncate the integral at some point. However, it is nice that one can
estimate the resulting error for each entry in the matrix when approximating the integral by

\[
\hat{TE}_{l_i,u_i}(T_j) := \int_0^{UB} TE_{l_i,u_i}(T_j) \{ M = m \} f_M(m) \, dm \\
+ \mathbb{P}(M > UB) \, TE_{l_i,u_i}(T_j) \{ M = UB \},
\]

i.e. assuming \( TE_{l_i,u_i}(T_j) \{ M = m \} \) to be constant in \( m \) above the upper bound \( UB \). This is possible as we know that this function is increasing in \( m \) with upper bound \( u_i - l_i \) and thus

\[
|\hat{TE}_{l_i,u_i}(T_j) - TE_{l_i,u_i}(T_j)| \\
\leq \mathbb{P}(M > UB) \, (u_i - l_i - TE_{l_i,u_i}(T_j) \{ M = UB \}).
\]

As a consequence, we are able to choose \( UB \) according to a specific error bound if required.

2.1 Comparison of base correlations In the present paragraph, the base correlations of the inhomogeneous model are compared with the base correlations using the LHP assumptions, which were computed in Vol. I. For a detailed description of the market data used, the interested reader is referred to that document. Now, the marginal default probabilities are chosen such that the market quotes for ten year single name CDS contracts are matched. The results can be found in Figure 1. One can observe that the general level of base \( \alpha \)'s is considerably higher in the inhomogeneous model. There is an
intuitive explanation for that. Compared to the homogeneous case, the value of all base tranches is increased when considering an inhomogeneous model with the same average default probability. This is due to the fact that base tranches essentially equal a basket of $k$th-to-default swaps, including $k = 1$ until $k = n$ for some $n$. The value of those swaps is considerably increased in the inhomogeneous case as, e.g., a first-to-default swap on a portfolio of one very risky and nine almost riskless companies is worth more than a first-to-default swap on a homogeneous pool of ten companies with smaller default risk, as it is basically a protection against the default of the most risky company. Since the value of all base tranches is higher in the inhomogeneous case, one needs a higher correlation to match the market price, as the value of those tranches is decreasing in the correlation parameter.

2.2 Tranche sensitivity with respect to the correlation

In the present paragraph, the sensitivity of the value of the different tranches with respect to changes of the correlation parameter $\alpha$ is investigated. The investigation is carried out with the data of one day only, here November 16, 2012 is investigated. Values of $\alpha$ between 0 and 0.9 are considered and the result can be found in Figure 2. It is very interesting that the behaviour of the tranche value with changing correlation can differ between the LHP case and the inhomogeneous case as can be seen here for tranches 3 and 4. Furthermore, considering the $\alpha$ with the best overall fit (error measured in terms of the sum of squared tranche values), one can observe that in the inhomogenous model the error is reduced by 50%. As expected, taking into account more information about the underlying structure allows for a better fit.

2.3 Tranche sensitivity with respect to single name CDS

In a last paragraph, the sensitivity of a tranche value with respect to changes in the corresponding single name CDS is investigated. Such investigations are the main reason for considering inhomogeneous models, as one might be interested in analysing the impact of different possible scenarios. Furthermore, the results are interesting for the purpose of hedging. Here, we exemplarily investigate the equity tranche. The correlation parameter $\alpha$ is set to the base correlation for the first tranche. For simplicity, we consider

$$\frac{\partial}{\partial s_i} V_{L,u_1}$$

i.e. the derivative of the value of the equity tranche $V_{L,u_1}$ with respect to the running spread $s_i$ of a single name CDS, which we will approximate with the absolute change in value when increasing the running spread by 1 bp. The result can be found in Figure 3, where it is additionally compared to the deltas for a different value of $\alpha$. One can observe that the delta is increasing in the spread as expected, and that the steepness of the curve depends on the value of the correlation parameter $\alpha$. Repeating the investigation for other tranches than base tranches reveals that the monotonicity is destroyed in those cases.
3 Further extensions

There is obviously still a lot of room for relaxations of the previously made assumptions. In this section, we shortly want to introduce the necessary ideas and comment on possible implementations without a detailed analysis. In a first step, the assumption of equal weights and equal and deterministic recovery rates is relaxed. One might allow the recovery rates to depend on the common factor $M$, or at least to be inhomogeneous. This can be realized almost analogously to our previous procedure. Again, one will consider everything conditioned on the common
Fig. 3: The deltas of the equity tranche with respect to the running spread of one single name CDS, plotted versus the corresponding running spread. Since the considered portfolio consists of 125 single names, we end up with 125 different deltas, which are all visualized. The investigation is conducted for two different values of $\alpha$, the actual base correlation and an arbitrarily chosen value.

The distribution of this object can be approximated using the second algorithm presented in Hull, White (2004). It is based on the idea of bucketing the loss distribution, i.e. one has to define $K$ buckets $B_0 := [0, b_0), B_1 := [b_0, b_1), \ldots, B_{K-1} := [b_{K-1}, \infty)$ (actually, as we considered a normalized portfolio, we could have chosen 1 instead of $\infty$). We will then compute the $K$ probabilities $\mathbb{P}(L^{(d)}_t \in B_i | M = m)$ and also $A_i := \mathbb{E}[L^{(d)}_t | L^{(d)}_t \in B_i, M = m]$, i.e. the expected loss for each bucket. These quantities will be computed by one-by-one adding the different constituents to the portfolio (starting from an empty portfolio) and updating the required quantities accordingly. This will serve as an approxima-
tion of the true loss distribution and using this approach, again integrating over the density of $M$, allows to price CDOs in this context.

If one additionally wants to make the dependence structure inhomogeneous, a possible idea would be to replace the Gumbel copula by a hierarchical Gumbel copula. By that, one can increase the dependence in between groups (e.g. the industry sectors) while keeping the dependence between companies of different groups small. A more detailed introduction into the usage of those dependence structures for CDO pricing can be found in Hofert, Scherer (2011). However, this approach requires the usage of Monte Carlo techniques, making it slower and more difficult to calibrate. It can be sped up significantly by putting some effort into the implementation using importance sampling or other techniques, but a description of those techniques lies outside the scope of this short note.

4 Conclusion

In the present note, we extended the CDO pricing model presented in Vol. I to non-homogeneous portfolios. The extension to finite portfolios with inhomogeneous marginals was investigated in more detail, explaining the impact on pricing and sensitivities. Furthermore, other ways of relaxing the simplifying assumptions were sketched.

References

