ON CONVEXITY 
ADJUSTMENTS FOR STOCK 
DERIVATIVES DUE TO 
STOCHASTIC REPO MARGINS

German Bernhart
XAIA Investment GmbH
Sonnenstraße 19, 80331 München, Germany
german.bernhart@xaia.com

Jan-Frederik Mai
XAIA Investment GmbH
Sonnenstraße 19, 80331 München, Germany
jan-frederik.mai@xaia.com

December 16, 2013

Abstract
Repo transactions are of high relevance for functioning financial markets, among others as they are a necessary tool for short selling. Short selling in turn is necessary for hedging, in particular for the hedging of stock derivatives. Most mathematical models in this context assume the repo margins to be constant for reasons of tractability. However, in reality those rates are stochastic and highly correlated with the development of the underlying stock. The aim of the present article is to investigate the impact of this feature on stock derivative pricing, respectively the error caused by ignoring this effect. For this purpose, it is shown how to incorporate a reasonable stochastic model for the repo margin into a stock price process. We adopt an idea from 1.5-factor credit-equity models to create a reciprocal relationship between stock price and repo margin, as observed in the marketplace. The result is a convexity adjustment for all kind of stock derivatives, which is ignored when the repo margin is modeled constantly. A concrete formula for the density of the stock's log-return can be found within a repo-enhanced Black-Scholes model. Similar as in the case of Asian option pricing, the involved mathematical techniques rely on well-known results for Yor's process and the Hartman-Watson law.

1 Motivation
When possessing a stock, it is possible to lend it, and the borrower of a stock implicitly pays a fee to the lender called repo margin. Borrowing a stock is the technical way of how people short-sell the stock in the marketplace. Shortselling the stock is often necessary for derivative dealers, because in the classical arbitrage pricing theory negative hedge ratios for stock derivatives are quite common. The repo margin must therefore be viewed as a market mechanism controlling the supply and demand for short-selling. Indeed, it is empirically observed in the marketplace that the repo margin is significant for distressed stocks, because the demand for short-selling increases both due to the advent of speculative traders and due to the requirement of derivative desks to expand their delta hedges. For this reason, it makes sense to model the repo margin stochastically and in a reciprocal relationship to the stock price. The present article investigates the quantitative effect of this phenomenon by propo-
sing a tractable modeling approach in the Black-Scholes model and demonstrating the impact on derivatives prices.

1.1 What is a repo agreement? There are several transaction types that are economically equivalent to repo-agreements but differ in terms of the legal structure. All these will be summarized under the term repo-agreement in the present paper. A simplified, stylized repo-agreement is assumed to work as follows: assume party A possesses the stock and lends it to party B over the time interval \([t, T]\). Then party B receives the stock with value \(S_t\) at time \(t\) and gives the cash amount \(S_t\), called the collateral, to party A\(^1\). At time \(T\), party A gets her stock back from B, and has to pay back the collateral with interest to party B. The interest rate that party A has to pay on the collateral, called repo rate, is smaller than the risk-free interest rate (otherwise she wouldn’t lend the stock) and denoted by \(r_{t,T} - \delta_{t,T}\), where \(\delta_{t,T}\) is called repo margin. Since party A can invest the collateral at the risk-free rate \(r_{t,T}\) during \([t, T]\) and only has to pay the smaller repo rate, one may conclude that possessing the stock earns the repo margin \(\delta_{t,T}\). Conversely, it can be interpreted as a fee paid for borrowing the stock. Clearly, party A has an incentive to lend the stock to party B, because she can earn the repo margin. However, there are several factors that can keep market participants from supplying their collateral to the repo markets which are described in Duffie (1996). Duffie (1996) also shows why \(\delta_{t,T} \geq 0\) has to hold. There are no other restrictions and \(\delta_{t,T}\) is basically defined by supply and demand. Repo contracts in the bond markets, in particular in US treasuries, are well investigated in the literature, see, e.g., Jordan, Jordan (1996), Duffie (1996). Duffie (1996) analyzes why certain instruments “trade special”, i.e. with a higher repo margin than other instruments of the same issuer. He identifies several factors influencing this effect. In the equity markets, there is only one instrument of an issuer, so “trading special” means a repo margin considerably above the average repo margin. D’Avolio (2002) empirically describes the market for borrowing stock and identifies disagreement among investors as one of the factors indicating “specialness”. Indeed, it is observed in the marketplace that the repo margin is typically very high for distressed names, and moderate for stocks in a calm market environment. To provide numeric examples, repo margins such as 50 bps are quite common in calm markets, while for more distressed names repo margins between 3% and 7% are observed frequently\(^2\). Instead of explaining the impact of short selling-fees on the stock price via equilibrium models as it is done in, e.g., Duffie et al. (2002), here we are concerned with the impact of repo rate behavior on derivative pricing when starting from a model for stock price and repo margin.

1.2 What is the use of repos? There are many functions of repo agreements, see, e.g., ICMA (2013) for an overview. Here, we want to focus on the short-

\(^1\)During the lending period \([t, T]\), party B pays all dividend cash flows net of withholding tax directly to party A.

\(^2\)A recent example for an extreme repo margin is IVG for which the quote was at a ridiculous level of 75% prior to its default.
selling aspect. Assume that party B wants to short-sell the stock to a third party at time $t$, for example with a speculative idea or for some hedging purpose. Then she can borrow the stock from party A and sell it to the third party. At time $T$, party B must give the stock back to party A. For this purpose, party B must buy the stock at time $T$ from someone else at the price $S_T$. In return, she receives the cash payment $S_t \exp(r_{t,T} - \delta_{t,T})$ from party A. Hence, at time $T$ party B receives the cash flow $S_t \exp(r_{t,T} - \delta_{t,T}) - S_T$, which is a payoff that is positive whenever the stock has lost enough value between $[t, T]$, so it is an investment that is “short equity”. One also observes that the payoff is decreasing in the repo margin $\delta_{t,T}$, emphasizing that the repo margin should actually be thought of as a fee for short-selling.

The short-selling function of repo contracts gives an explanation for high repo margins for distressed stocks as mentioned before. The reason is that the demand for speculative “short equity” investments increases when the market believes in a downturn of the stock, i.e. in a distressed environment. Furthermore, after a market downturn traders short put options might have to increase their negative hedging delta. In those cases, the repo desks must increase their repo margins in order to match the supply and demand orders in their books. In particular, for modeling tasks this implies that the repo margin should exhibit a reciprocal relationship with the stock price, respectively creditworthiness, of the stock-issuing company. Unfortunately, the existing literature mainly focuses on the impact of the shorting market on future stock returns, see e.g. Cohen et al. (2007) and the references therein, and not on the empirical relation between returns and repo margins. However, this reciprocal relationship is easy to argue for and well-known in the market. Modeling this effect and investigating its impact on derivative pricing is the major incentive of the present article.

1.3 Why should one model it?

It has been shown in the previous paragraph that the repo margin should be thought of as a fee for short-selling a stock. This holds regardless of the instruments used, as it is of course also considered when pricing a forward contract. In the current low interest rate environment, the impact of repo margins on forward prices is significantly more severe than the effect of the interest rate term structure. When selling a put option and hedging via a short position in the stock, this fee has to be paid and consequently to be taken into account when pricing the put option. However, as the hedge ratio usually fluctuates, the risk of higher hedging costs over the horizon of the derivative should not be neglected. In particular when using the reciprocal relationship between stock price and repo margin, since this assumes that higher hedging costs occur in scenarios where the hedge has to be increased. So called “capital structure arbitrage” strategies are another example where this effect can be important. Hedging long-dated credit products by shorter-dated equity derivatives can become expensive when the hedge has to be prolonged but the repo margin increases. Ignoring the stochastic nature of repo margins

\[\text{Unfortunately, repo dealers are very secretive with respect to their data, so we cannot provide an exemplary graph of repo margins vs. stock price.}\]
would therefore underestimate the true hedging costs. For many stocks, this effect is secondary and can be ignored as the repo margin is almost unchanged and independent of the stock price movement. In some cases, in particular for distressed stocks as mentioned before, it can play an important role. The aim of the present paper is to estimate this effect on derivative prices in those cases.

2 Arbitrage pricing theory

In the following, interest rates are assumed to be deterministic and will be modeled by the instantaneous short rate \( \{ r_t \}_{t \geq 0} \). We will ignore dividends though it will be seen later how continuous dividend yields can be included easily. In fact, the modeling approach we take resembles very much the modeling of a continuous dividend yield. Instead of modeling the repo margin for discrete time periods \([t, T]\), we want to consider an instantaneous repo margin \( \{ \delta_t \}_{t \geq 0} \) analogously to the instantaneous short rate. Consequently, \( \delta_t \) denotes the repo margin for an instantaneous repo agreement at \( t \) and thus the instantaneous rate one can earn by lending out the stock. Assuming the stock to be constant over an interval \([t, T]\), continuously lending out the stock would earn
\[
S_t \left( \exp \left( \int_t^T \delta_s \, ds \right) - 1 \right).
\]
This is a mathematically convenient abstraction from the actual market practice of “open” or “continuing” lending agreements as described in Duffie et al. (2002) which are renewed each day.

We start from a very simple and well understood model, the Black-Scholes model, as this makes an interpretation of the results easier and helps to isolate the effects of the stochastic repo rate. This is in line with the explorative nature of the present paper. Extended investigations of the considered problem using more realistic model frameworks are postponed to further research, which the authors hope to trigger with the present note.

Given a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) with filtration \(\{\mathcal{F}_t\}\) modeling the market’s information flow and \(\mathbb{P}\) denoting the physical measure, we assume the stock price process to be modeled by a geometric Brownian motion with drift \( \mu \) and volatility \( \sigma > 0 \), i.e. by the solution of the SDE
\[
dS_t = S_t (\mu \, dt + \sigma \, dW_t),
\]
with given \( S_0 \) and \( \{W_t\}_{t \geq 0} \) a Brownian motion under \( \mathbb{P} \). As usual in the context of dividends or repos, for no arbitrage arguments one has to consider \( \{Z_t\}_{t \geq 0} \), the wealth-process resulting from investing in the stock \( S \). Assuming that the earnings from lending out the stock are reinvested into the stock, we can state as SDE for \( Z \)
\[
dZ_t = Z_t \delta_t \, dt + Z_t (\mu \, dt + \sigma \, dW_t),
\]
with \( Z_0 = S_0 \). From this, \( Z_t = S_t \exp(\int_0^t \delta_s \, ds) \) follows easily. From the usual arbitrage pricing theory, we know that there exists an equivalent pricing measure \( \mathbb{Q} \) such that under \( \mathbb{Q} \), the discounted wealth process \( Y_t := \exp(-\int_0^t r_s \, ds) Z_t \) behaves like a geometric Brownian motion with zero drift and volatility \( \sigma \). Consequently,
\[
Z_t = \exp \left( \int_0^t r_s \, ds \right) Y_t = S_0 \exp \left( \int_0^t r_s \, ds - \frac{t \sigma^2}{2} + \sigma \dot{W}_t \right),
\]
with \( \{W_t\}_{t \geq 0} \) a Brownian motion under \( Q \), and

\[
S_t = \exp \left( \int_0^t (r_s - \delta_s) ds \right) Y_t.
\]

This is the formula we want to work with in the following. As a side remark, one may also interpret \( \{\delta_t\} \) as a continuous dividend yield. Alternatively, it may also be interpreted as a combination of repo margin and dividend yield. We focus on the interpretation of a repo margin because this is the relevant quantity for highly distressed stocks, which often do not pay dividends in the near future anyway.

2.1 Stochastic modeling approach for the repo margin

Since most practical tasks require one-factor models, we assume that the repo margin \( \{\delta_t\} \) is defined via \( \delta_t := g(Y_t) \) for a decreasing function \( g \). The reason for this ansatz is that it provides us with the desired reciprocal relationship of the repo margin and the company’s creditworthiness: when \( \{Y_t\} \), and thus \( \{S_t\} \), evolves better than expected, the repo margin decreases and vice versa, with the boundary assumptions

\[
(a) \quad \lim_{x \downarrow 0} g(x) = \infty, \quad (b) \quad \lim_{x \uparrow \infty} g(x) = 0.
\]

Assumption (a) means “infinite demand/zero supply for short-selling when stock value is approaching zero” and assumption (b) means “no short-selling restrictions when the stock price is very high”.

A simpler modeling approach – often applied in the industry – more simplistically assumes that \( \delta_t \) is a constant. However, this ansatz ignores supply and demand effects of the short-selling market during the considered modeling horizon \([t, T]\). For example, assume that we evaluate the price of a European call option with maturity \( T \). If the repo margin \( \delta_t \) is a constant, the model price of the option will be decreasing in this value (because the drift of the stock price is negatively affected by the repo margin). Consequently, the choice of model for \( Y_t \) and \( g \) within our more realistic modeling approach might induce “convexity adjustments” to the call option price and option prices in general due to the fact that \( \delta_t \) becomes a random variable.

3 A concrete formula within the Black-Scholes cosmos

We provide a concrete modeling approach for the stock and repo margin in our “repo-enhanced Black-Scholes model” in the sequel. We assume that

\[
\delta_t = g(Y_t) = \delta_0 \left( \frac{Y_0}{Y_t} \right)^p, \quad t \geq 0,
\]

for modeling constants \( \delta_0, p \geq 0 \). This parametric choice for \( g \) induces the desired reciprocal relationship and is inspired by an idea in 1.5-factor credit-equity modeling, where the company’s default intensity is linked to the company’s stock price by a similar function, see, e.g., Carr, Linetsky (2006); Linetsky (2006). Observing an actual repo margin \( \delta_{0,T_1} \) in the marketplace for some maturity \( T_1 \), one can choose \( \delta_0 \) accordingly, i.e. such that

\[
S_0 \exp(r_{0,T_1} - \delta_{0,T_1}) = \mathbb{E}[S_{T_1}].
\]
This corresponds to
\[ S_0 \exp(-\delta_{0,T_1}) = \mathbb{E} \left[ Y_{T_1} \exp \left( -\int_0^{T_1} \delta_s ds \right) \right]. \]

But, of course, other choices for \( \delta_0 \) or even for the whole function \( g \) are imaginable as well, e.g.
\[
\delta_0 := \delta_{0,T_1} \int_0^{T_1} \mathbb{E}[(Y_0/Y_t)^p] dt = \frac{1}{2} \sigma^2 p (1 + p) \delta_{0,T_1} \frac{1}{\exp(\frac{1}{2} \sigma^2 p (1 + p) T_1) - 1},
\]
which would be a good rule of thumb that implies that the expected value of \( \int_0^{T_1} \delta_s ds \) equals the observed quantity \( \delta_{t,T} \). The following theorem is the main contribution of the present article and provides an explicit expression for the density \( f_t \) of the log-return \( X_t := \log(S_t/S_0) - \int_0^t r_s ds \).

**Theorem 3.1 (Convexity adjustment on the density)**

Denoting by \( n(\cdot; \mu, \sigma^2) \) the density of the normal law with mean \( \mu \) and variance \( \sigma^2 \), we have
\[
f_t(x) = n(x; -\sigma^2 t/2, \sigma^2 t) + \frac{p}{2} e^{-\frac{x^2}{2}} \times \\ \times \int_0^\infty \left\{ \exp \left( \frac{1 + e^{-x p} - \frac{2u \delta_0}{p^2 \sigma^2}}{-2u} \right) \theta \left( \frac{e^{-\frac{x p}{2}} - \frac{2u \delta_0}{p^2 \sigma^2}}{u}, \frac{\sigma^2 p^2 t}{4} \right) - \exp \left( \frac{1 + e^{-x p}}{-2u} \right) \theta \left( \frac{e^{-\frac{x p}{2}}}{u}, \frac{\sigma^2 p^2 t}{4} \right) \right\} \frac{1}{u} du,
\]
where
\[
\theta(r, w) := \frac{r e^{\frac{p w}{2}}}{\sqrt{2 \pi w}} \int_0^\infty e^{-\frac{y^2}{2w}} \cosh(y) \sinh(y) \sin \left( \frac{\pi y}{w} \right) dy.
\]

**Proof**

Postponed to the Appendix.

The density of the log-return in Theorem 3.1 is the sum of two parts. The first part is simply \( n(x; -\sigma^2 t/2, \sigma^2 t) \), which would be the density of the log-return in the classical Black-Scholes setup without repo-adjustment, i.e. when \( \delta_0 = 0 \). The second, numerically more challenging term must therefore integrate to zero (i.e. must have negative and positive parts), and can be seen as a repo-induced convexity adjustment to the density in the standard case. Regarding the numerical computation of \( f_t(x) \), we provide some advice in the Appendix.

Figure 1 visualizes the density of Theorem 3.1 in comparison with the normal density which the log-return would have if the repo margin was modeled by the associated constant \( \delta_{0,t} \). This can be computed using \( \delta_{0,t} := -\log(\int_0^\infty \exp(x) f_t(x) dx) \). The considered artificial example represents a company “trading special”, i.e. with a high repo margin, and with a strong link between stock price and repo margin as one would model it for a distressed name. It is observed that the repo-adjustment induces a significant skew to the log-return distribution.

Figure 2 visualizes the effect of this skew on the pricing of derivatives. Here, implied volatilities of European put options are
considered for different values of $p$, which determines how strong the relationship between stock and repo margin is. The implied volatilities are computed by inverting the Black-Scholes formula with interest rate adjusted by the constant repo margin $\delta_{0,t}$ in order not to compare apples (model prices with repo) and oranges (model prices without repo). One can observe two effects: To begin with, even for smaller values of $p$, the implied volatilities are increased compared to an implied volatility of 60% in the case of a constant repo margin. As argued before, considering a stochastic repo margin increases the price of options and hence the implied volatilities. This is the so called convexity adjustment one expected. The second effect is the form of this convexity adjustment. It can be seen from Figure 2 that it induces a volatility skew, which is the more pronounced the higher $p$ is chosen. Hence, for distressed names, a part of the volatility skew observed in the market may be attributed to the effect of stochastic repo margins. Furthermore, the change in implied volatilities is considerable even for moderate values of $p$, so ignoring the stochastic nature of repo margins for names “trading special” may have non-negligible effects, in particular for out-of-the-money puts.

4 Conclusion

It was argued that a stochastic model for the repo margin in a stock price model should exhibit a reciprocal relationship with the stock price itself. Having adopted an idea from commonly applied 1.5-factor credit-equity models, it was shown how this can be accomplished. Within a Black-Scholes model a concrete formula for the density of the log-return of a repo-adjusted stock price was derived. It was illustrated how the stochastic repo margin resulted in a skew of the otherwise normal log-return density and also in a skew of the implied Black-Scholes volatilities. The effect...
is considerable in particular for distressed names and out-of-the-money options. Furthermore, it shows that parts of the volatility skew for distressed names can be attributed to the stochastic nature of repo margins.

Appendix: Proof of Theorem 3.1

In the second equality of the computation to follow we apply the fact that \( \{ W_t \}_{t \geq 0} = \{ -W_t \}_{t \geq 0} \); in the third equality we apply the Brownian scaling property \( \{ W_{2p^2 t^4/4} \}_{t \geq 0} = \{ -W_{2p^2 t^4/4} \}_{t \geq 0} \); in the fourth equality we apply the substitution \( u = \sigma^2 p^2 s/4 \); and in the fifth equality we introduce the notation \( A_{t}^{(\nu)} := \int_{0}^{t} \exp (2 (W_s + \nu s)) \, ds \) for Yor’s process.

\[
\begin{align*}
\exp \left( -\int_{0}^{t} r_s \, ds \right) \frac{S_t}{S_0} &= \exp \left( -\frac{\sigma^2}{2} t + \sigma W_t - \delta_0 \int_{0}^{t} e^{\frac{p \sigma^2}{2} - p \sigma W_s} \, ds \right) \\
&= \exp \left( -\frac{\sigma^2}{2} t - \sigma W_t - \delta_0 \int_{0}^{t} e^{\frac{p \sigma^2}{2} + p \sigma W_s} \, ds \right) \\
&= \exp \left( -\frac{\sigma^2}{2} t - \frac{2}{p} W_s + \frac{2}{p} W_s - \delta_0 \int_{0}^{t} e^{\frac{p \sigma^2}{2} + 2W_s} \, ds \right) \\
&= \exp \left( -\frac{\sigma^2}{2} t - \frac{2}{p} W_s + \frac{2}{p} W_s - \frac{4 \delta_0}{\sigma^2 p^2} \int_{0}^{t} e^{\frac{p \sigma^2}{2} + 2W_s} \, ds \right) \\
&= \exp \left( -\frac{2}{p} \left( W_s + \frac{2}{p} W_s \right) - \frac{2}{\sigma^2 p^2} - \frac{2}{\sigma^2 p} \int_{0}^{t} e^{\frac{p \sigma^2}{2} + 2W_s} \, ds \right) \\
&= \exp \left( -\frac{2}{p} \left( W_s + \frac{2}{p} W_s \right) - \frac{2}{\sigma^2 p^2} - \frac{2}{\sigma^2 p} \int_{0}^{t} e^{\frac{p \sigma^2}{2} + 2W_s} \, ds \right).
\end{align*}
\]

We adopt the notation

\[
a_t(x, u) \, du := \mathbb{P} \left( \left. A_{t}^{(1/p)} \in du \right| W_t + \frac{1}{p} t = x \right)
\]
from Yor (1992) to proceed. Notice in particular that this conditional probability law is independent of \( p \), the precise formula for \( a_t(x, u) \) from (Yor, 1992, (6.c)) is applied later below. It is observed that the log-return \( X_t := \log(S_t/S_0) - \int_0^t r_s \, ds \) in concern consists of two parts:

\[
X_t = -\frac{2}{p} \left( W_{\sigma^2 t^{1/2}} + \frac{1}{p} \frac{\sigma^2 p^2 t}{4} \right) - \frac{4}{\sigma^2 p^2} \mathcal{A}^{(1/p)}_{\sigma^2 t^{1/2}},
\]

The first part \((*)\) has a normal distribution with mean \(-\sigma^2 t/2\) and variance \(\sigma^2 t\), which equals the log-return distribution in the classical Black-Scholes model without (stochastic) repo margin. Hence the second part \((**)\) can be viewed as a convexity adjustment due to the stochastic repo model. We proceed with computing the distribution function of \( X_t \), applying the standard notation \( \Phi \) for the standard normal distribution function.

\[
P(X_t \leq x) = \mathbb{P} \left( \mathcal{A}^{(1/p)}_{t/\sigma^2} \geq \frac{p^2 \sigma^2}{4 \delta_0} \left( -\frac{2}{p} (W_t + t/p) - x \right) \right)
= \mathbb{P} \left( -\frac{2}{p} (W_t + t/p) - x \leq 0 \right)
+ \mathbb{E} \left[ \left\{ -\frac{2}{p} (W_{t+t/p}) \right\} \right. \int_{-\infty}^{\infty} a_t(W_t + t/p, u) \, du \left. \right\}
= \mathbb{P} \left( -W_t \leq \frac{xp}{2\sqrt{t}} + \frac{\sqrt{t}}{p} \right)
+ \int_{-\infty}^{\frac{xp}{2\sqrt{t}}} \frac{1}{\sqrt{2\pi} t} e^{-\frac{(w-t/p)^2}{2t}} \int_{-\infty}^{\infty} a_t(w, u) \, du \, dw
= \Phi \left( \frac{xp}{2\sqrt{t}} + \frac{\sqrt{t}}{p} \right)
+ \int_0^{\infty} \int_{-\frac{xp}{2\sqrt{t}}}^{-\frac{\sqrt{t}}{p}} \frac{1}{\sqrt{2\pi} t} e^{-\frac{(w-t/p)^2}{2t}} a_t(w, u) \, dw \, du
= \Phi \left( \frac{xp}{2\sqrt{t}} + \frac{\sqrt{t}}{p} \right) + e^{-\frac{xp}{2p^2}} \times
\int_0^{\infty} \int_{-\frac{xp}{2\sqrt{t}}}^{-\frac{\sqrt{t}}{p}} \frac{1}{u} \exp \left( \frac{1 + e^{2w}}{-2u} + \frac{w}{p} \right) \theta \left( \frac{e^w}{u}, t \right) \, dw \, du,
\]

where the last equality makes use of (Yor, 1992, (6.c)). Finally, taking the derivative with respect to \( x \) in the last formula yields an expression for the density of \( X_t \):

\[
f_{\frac{X_t}{\sigma^2 t^{1/2}}} (x) = \frac{p}{2} \sqrt{\frac{8}{\pi}} \frac{e^{-\left(\frac{xp^2+2t}{8p^2t}\right)^2}}{\sqrt{2\pi} t} + \frac{p}{2} e^{-\frac{x}{2p^2 t^{1/2}}} \times
\int_0^{\infty} \left\{ \exp \left( \frac{1 + e^{-x} p - 2 u \delta_0}{-2u} - \frac{2 u \delta_0}{p^2 \sigma^2} \right) \theta \left( \frac{e^{-x} p}{u}, t \right) \right. \left. - \exp \left( \frac{1 + e^{-x} p}{-2u} \right) \theta \left( \frac{e^{-x} p}{u}, t \right) \right\} \frac{1}{u} \, du.
\]
Plugging in the value $t := p^2 \sigma^2 \tilde{t}/4$ yields the claimed formula for the desired density of $X_{\tilde{t}}$, with $\tilde{t} > 0$ arbitrary.

### Appendix: Numerical aspects when evaluating the Hartman-Watson density

In our formula for $f_t(x)$ one faces the numerically challenging task of evaluating the function $\theta(r, w)$. Denoting

$$I_\nu(z) := \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \nu + 1)} \left(\frac{z}{2}\right)^{m+\nu}$$

the modified Bessel function of the first kind, the function $w \mapsto \theta(r, w)/I_0(r)$, $w > 0$, is the density of a probability law, say $\mu_r$, on the positive half-axis, called the Hartman-Watson law. It was shown in Hartman (1976) that this law is infinitely divisible with Laplace transform given by $u \mapsto I_{\sqrt{2u}}(r)/I_0(r)$, $u \geq 0$. The numerical evaluation of the density of the Hartman-Watson law is a challenging task because the integrand in the formula for $\theta(r, w)$ is highly oscillating. For instance, the references Barrieu et al. (2004); Gerhold (2011) derive asymptotics for this distribution as the argument $w$ tends to zero and indicate that its numerical evaluation is of paramount interest when dealing with Asian options.

In order to circumvent some of these numerical challenges, we resort to Laplace inversion methods in order to derive $\theta(r, w)$ from $\{I_{\sqrt{2u}}(r)\}_{u \geq 0}$. We apply the Gaver-Stehfest algorithm, for a rigorous proof and a good explanation of this method see Kuznetsov (2013). In particular, it is not difficult to observe from Yor’s expression (1) for the Hartman-Watson density that (Kuznetsov, 2013, Theorem 1(iii)) applies, which justifies the approximation

$$\theta(r, t) \approx \frac{\log(2)}{t} \sum_{k=1}^{2n} a_k(n) I_{\sqrt{2k \log(2)/t}}(r),$$

for $n \in \mathbb{N}$ large enough, where for $j = 1, \ldots, 2n$ we have

$$a_k(n) = \frac{(-1)^{k+n}}{n!} \min\{k,n\} \sum_{j=\lfloor (k+1)/2 \rfloor} \binom{n}{j} \binom{2j}{j} \binom{j}{k-j}.$$

Plugging this formula into the formula for $f_t$ from Theorem 3.1, we end up with a numerically feasible expression for the convexity adjustment. In our MATLAB implementation for creating the plots in Figure 1 and Figure 2, we chose $n = 10$ and evaluated the integral by using trapezoidal integration after transforming the integration domain to $[0, 1]$ by substituting $v = -\log(u)$, which yielded reasonable results for the considered parameter space. Moreover, the required evaluations of the modified Bessel functions have been made using the MATLAB built-in function besseli. The results have been back-checked via a Monte Carlo simulation. For small $p < 0.2$ and very large $p > 1.75$ we run into numerical problems, while for $p \in [0.5, 1.5]$, the results are reliable. However, finding a numerically stable implementation of the presented formula for arbitrary parameter constellations is not easy. The authors confess that the formula of Theorem 3.1 is predominantly of educational value. Its efficient implementation, potential short-hand approximations of the convexity adjustment, or even extensions of the formula to situations outside the Black-Scholes cosmos, are postponed to further research, which the authors hope to trigger with the present note.
References


