CAP STRUC ARB HEDGING: THE DELTA-TO-JUMP RATIO

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Abstract Regarding the hedge of a capital structure arbitrage position, we focus on both the jump-to-default risk and the delta risk at the same time. In the earlier article Mai (2014) we pointed out that when a put option is used to hedge a long credit investment, not only the number of puts but also the strike price is determined by these two targets. Slightly reformulating this observation, in the present article we find that the essential criterion for the choice of strike price is that the delta-to-jump ratio of the put equals the delta-to-jump ratio of the long investment, thereby uniting both targets in a single quantity. If one uses two short instruments for hedging, for instance a shorter-dated CDS and a put or two puts with different strikes, then it is important to make sure that the delta-to-jump ratio of the long investment lies in between the delta-to-jump ratios of the two hedging instruments. The appropriate hedge can then conveniently be interpreted in terms of a convex combination of the two delta-to-jump ratios of the hedging instruments.

1 Introduction

Assuming we are exposed to a long investment in some company XY, modeled as a function depending on the current stock price of XY, there are two essential sources of risk that we keep an eye on. On the one hand, there is what we call the *jump-to-default risk*, which is the risk to experience a severe and sudden market price change due to a (news-driven) change in credit-worthiness. On the other hand, there is the daily mark-to-market risk, which we also call the *delta risk*. This is the risk of regular and small market price changes due to daily fluctuations in supply and demand. We quantify the jump-to-default downside risk in terms of the amount of money that is lost in case of an instantaneous bankruptcy, often depending on recovery rate assumptions. The delta risk is always quantified using some "reasonable" stochastic model, by computing the first derivative of the long investment in concern with respect to the current stock price level.

When seeking to hedge away both sources of risk, these two different quantitative measures need to be taken into account at the same time. Loosely speaking, in order to eliminate both sources of risk at the same time, in general one expects to require two degrees of freedom in one's hedging instrument(s). Indeed, our earlier article Mai (2014) shows that when hedging a long credit investment with a single put option, the two dimensions "number of options" and "choice of strike price" are uniquely determined by the aforementioned two target criteria "jump-to-default" and "delta". Unfortunately, however, the so determined strike price of the option is sometimes not very liquid in the marketplace. As an alternative in such a situation the present article investigates

when two different hedging instruments, e.g. put options with two different strike prices, can be applied in order to allocate a part of the hedge to more liquid hedging instruments.
As a first technical step it turns out convenient to introduce the so-called *delta-to-jump ratio* for each involved asset, which is simply the quotient of the two aforementioned quantitative measurements, namely the delta divided by the jump-exposure. The result of Mai (2014) may then be reformulated by observing that the delta-to-jump ratio of the long credit investment in concern must be equal to the delta-to-jump ratio of the hedging instrument (which is a put option). When option maturity is fixed, this single criterion determines the strike price to be used uniquely. Now if we use two hedging instruments, we show that it is important to choose one instrument with delta-to-jump ratio larger than that of the long investment and the other with a delta-to-jump ratio smaller. Intuitively, the delta-to-jump ratio of the considered long credit investment is a convex combination of the two delta-to-jump ratios of the hedging instruments, and an appropriate hedge can be interpreted in terms of this convex combination.
The remaining article is organized as follows. Section 2 introduces required notions and algebraically solves the aforementioned hedging problem. Section 3 illustrates the resulting methodology in a practical example.

2 The algebra

We denote by L the current market price of a long investment with unit nominal. Typically, L is either the price of a bond or the negative value of the upfront of a credit default swap (CDS), i.e. $(-L)$ denotes the CDS upfront value. However, it could more generally also be a portfolio of bonds/ CDS/ equity investments related to XY with a net long exposure. We denote by ΔL the Delta of L with respect to the current stock price S_0, that is

$$\Delta L = \frac{\partial}{\partial S_0} L(S_0).$$

Notice that this quantity is model-dependent, and must be computed within some credit-equity model, see Mai (2012) for general background on the latter. Furthermore, we denote by ΦL the expected loss of the long investment L in case of an immediate default event. For instance, if L denotes a bond price and we denote by R a recovery rate assumption, $\Phi L = L - R$. Similarly, if $(-L)$ denotes a CDS upfront, $\Phi L = 1 - R - (-L)$. The long investment L exhibits two sources of risk, namely delta risk and jump-to-default risk, measured in terms of ΔL and ΦL. We call the quotient $\Delta L/\Phi L$ the *delta-to-jump ratio* of L. Notice that it is well-defined, since the denominator is positive, and positive, since the delta is positive (long investment).
Our goal is to find a hedging position that eliminates both sources of risk. To this end, we denote by H_1 and H_2 two different short investments used for hedging L. Concretely, H_2 is always assumed to be the price of a put option on the stock price of the company with some strike K_2. And H_1 might either be another put option with smaller strike $K_1 < K_2$, or it might be the upfront price of a CDS on the company. In either case, we denote again by ΔH_i the delta of H_i with respect to the current stock price S_0,

i.e.

$$\Delta H_i = \frac{\partial}{\partial S_0} H_i(S_0), \quad i = 1, 2.$$

Similarly, we denote by ΦH_i the expected gain of H_i in case of an immediate default event. This means that $\Phi H_2 = K_2 - H_2$ and

$$\Phi H_1 = \begin{cases} K_1 - H_1 & \text{, if } H_1 \text{ is a put price} \\ 1 - R - H_1 & \text{, if } H_1 \text{ is a CDS upfront} \end{cases},$$

where again R denotes a recovery assumption on the CDS in case of an immediate default. Notice in particular that this definition of ΦH_2 (and also of ΦH_1 in case of H_1 being a put option) assumes that the stock price equals zero in case of an instantaneous bankruptcy, which is a common modeling assumption that is justified for heavily indebted companies. We call the quotient $-\Delta H_i / \Phi H_i$ the *delta-to-jump ratio* of the hedging instrument H_i, noticing again that this is well-defined (since ΦH_i is positive) and positive (since ΔH_i is negative, as it is a short instruments). We need to make the following technical assumption:

$$-\frac{\Delta H_2}{\Phi H_2} > -\frac{\Delta H_1}{\Phi H_1}. \tag{1}$$

In particular, (1) makes clear that H_1 cannot be the same instrument as H_2, i.e. we truly consider two different hedging instruments. In words, (1) says that the delta-to-jump ratio of H_2 must be strictly larger than that of H_1. We will see below that this condition is necessary and sufficient to be able to eliminate both sources of risk. However, it is not sufficient to achieve this goal with long investments in both short instruments H_1 and H_2, which is what's desired. Notice that if we short one of H_1 or H_2, then essentially we increase our total long risk exposure given by L further. But we purposely wish to restrict ourselves to cases in which the investments in both H_1 and H_2 reduce our total long risk exposure given by L. In the following, we will explain that a necessary and sufficient condition for this desired situation to prevail is that

$$-\frac{\Delta H_2}{\Phi H_2} > \frac{\Delta L}{\Phi L} > -\frac{\Delta H_1}{\Phi H_1}, \tag{2}$$

which is an even stronger condition than (1). To explain this, denoting by N the given nominal of the long investment L, by x_2 the number of put options H_2, and by x_1 either the number of put options H_1 or - in case H_1 is a CDS - the nominal of H_1. We seek to find x_1, x_2 such that

$$\begin{aligned} &N\,\Phi L = x_1\,\Phi H_1 + x_2\,\Phi H_2 && \text{“jump-neutrality”} \\ &N\,\Delta L + x_1\,\Delta H_1 + x_2\,\Delta H_2 = 0 && \text{“delta-neutrality”.} \end{aligned}$$

Apparently, this is a linear equation system in two unknowns with the following unique solution under our assumption (1), as can readily be checked:

$$x_1 = \Big(\frac{N\,\Phi L}{\Phi H_1}\Big) \frac{-\frac{\Delta H_2}{\Phi H_2} - \frac{\Delta L}{\Phi L}}{-\frac{\Delta H_2}{\Phi H_2} - \big(-\frac{\Delta H_1}{\Phi H_1}\big)},$$

$$x_2 = \Big(\frac{N\,\Phi L}{\Phi H_2}\Big) \frac{\frac{\Delta L}{\Phi L} - \big(-\frac{\Delta H_1}{\Phi H_1}\big)}{-\frac{\Delta H_2}{\Phi H_2} - \big(-\frac{\Delta H_1}{\Phi H_1}\big)}.$$

It is further not difficult to verify that these formulas imply that the desired condition $x_1, x_2 > 0$ is equivalent to (2). Let us add a brief discussion about the intuition behind this optimal hedge.

Remark 2.1 (Interpretation and generalization)
On the one hand, the condition (2) is equivalent to the existence of a number $\epsilon \in (0,1)$ that represents the delta-to-jump ratio of L as a convex combination between the delta-to-jump ratios of the two hedging instruments, i.e.

$$\epsilon\Big(-\frac{\Delta H_1}{\Phi H_1}\Big) + (1-\epsilon)\Big(-\frac{\Delta H_2}{\Phi H_2}\Big) = \frac{\Delta L}{\Phi L}. \tag{3}$$

On the other hand, if only a single put option H was used to hedge the jump-to-default risk of L, the required number of put options x in this case is easily verified to be $x = N\,\Phi L/\Phi H$. In words, x times the jump-gain of H must be equal to the jump exposure of L. Only if the delta-to-jump ratios of L and H coincide, this hedge would also eliminate the delta-risk. Solving (3) for ϵ and comparing with the optimal solution, we observe that the optimal hedge x_1, x_2 is given by

$$x_1 = \epsilon\,\frac{N\,\Phi L}{\Phi H_1}, \quad x_2 = (1-\epsilon)\,\frac{N\,\Phi L}{\Phi H_2}.$$

In words, this means that ϵ (resp. $1-\epsilon$) represents a fraction of the hedge amount $N\,\Phi L/\Phi H_1$ (resp. $N\,\Phi L/\Phi H_2$) that would be required in case H_1 (resp. H_2) was used as the only hedging instrument to eliminate the jump-to-default risk of L. As a final remark, the very same logic can readily be generalized to the case of arbitrarily many hedging instruments: if $H_1, \ldots, H_n$ are hedging instruments for L, such that it is possible to find $\epsilon_1, \ldots, \epsilon_n \in [0,1]$ that imply the convex combination

$$\frac{\Delta L}{\Phi L} = \sum_{i=1}^{n} \epsilon_i \Big(-\frac{\Delta H_i}{\Phi H_i}\Big),$$

then the portfolio $N\,L + x_1\,H_1 + \ldots + x_n\,H_n$ is both jump-neutral and delta-neutral, where

$$x_i = \epsilon_i\,\frac{N\,\Phi L}{\Phi H_i}, \quad i = 1, \ldots, n.$$

The choice of $\epsilon_1, \ldots, \epsilon_n$ needs not be unique, but might be further optimized, for instance according to available market prices. Only in the presented case $n = 2$ it is uniquely determined, as shown.

3 Illustration

We have seen in the previous paragraph that the delta-to-jump ratio plays a fundamental role with regards to our hedging task. Consequently, the following observations are of crucial importance to understand what kind of hedging instruments are reasonable candidates:

(i) The delta-to-jump ratio $-\Delta H/\Phi H$ of a put option H that is not too deep in-the-money is typically increasing in the strike price[1].

[1] This statement relies on empirical evidence and moderate maturities, but is wrong in general. In the presented example it holds, and in case of the Black-Scholes model one can show that it certainly holds true for OTM strike prices, see the Appendix.

(ii) The delta-to-jump ratio $-\Delta H/\Phi H$ of a CDS upfront H is increasing in its maturity.

According to (i), there is a unique strike price K such that the delta-to-jump ratios of L and $H = H(K)$ coincide. This is precisely the strike level K that needs to be used in order to eliminate both jump-to-default risk and delta risk with a single put option as hedging instrument, according to Mai (2014). When using two puts with different strikes, condition (2) says that the strike K_1 needs to be smaller than K and K_2 needs to be larger than K. In fact, for any choice of $K_1 < K < K_2$ there is a desired solution.

If L describes a short CDS investment and H_1 is the upfront of a CDS with maturity shorter than that of L, then (ii) implies that $\Delta L/\Phi L > -\Delta H_1/\Phi H_1$. In this situation, in order to satisfy condition (2) necessarily the strike price K_2 must be chosen larger than the aforementioned strike price K. Reformulating this observation turns it into a quite intuitive statement: hedging an outright short CDS position with a single put option requires a smaller strike price than hedging a CDS flattener position with a single put option.

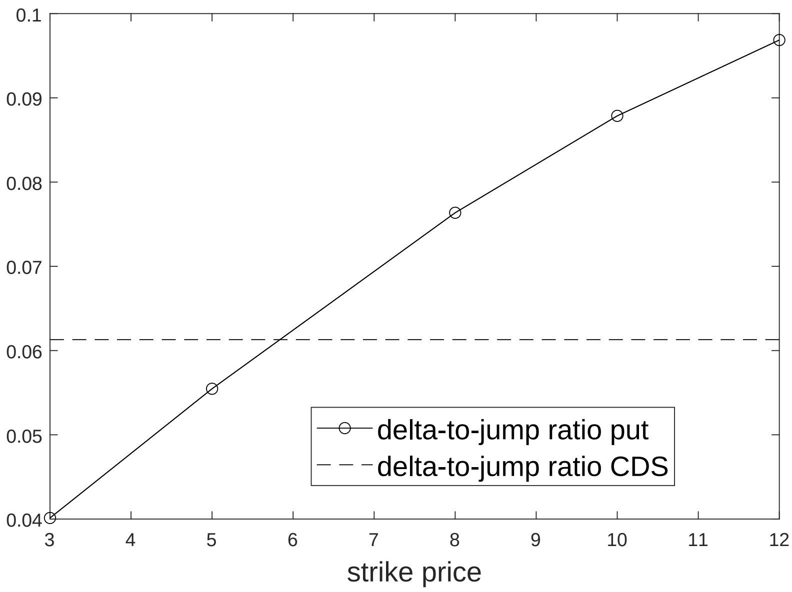

Fig. 1: The delta-to-jump ratio of a put option in dependence on its strike price, for the available strikes $3, 5, 8, 10, 12$.

To provide a concrete example, on 20 July 2020 we consider as long investment L a short CDS with maturity in June 2025 and 10 million USD nominal, at a price of 30% upfront with 500 bps running coupon, i.e. $L = -0.3$. We assume a recovery rate assumption of $R = 30\%$ to compute $\Phi L = 1 - R - (-L) = 0.4$. As hedging instruments we consider put options with maturity in January 2022 and different available strike prices, namely $3, 5, 8, 10, 12$. Figure 1 visualizes the delta-to-jump ratio of the available put options in dependence on the strike price. Apparently, this ratio is increasing in the strike price, thus confirming (i). The dotted line illustrates the delta-to-jump ratio of the long investment L. It is observed that the strike level K that equates delta-to-jump ratios of L and the put option lies between 5 and 8 in the current example.

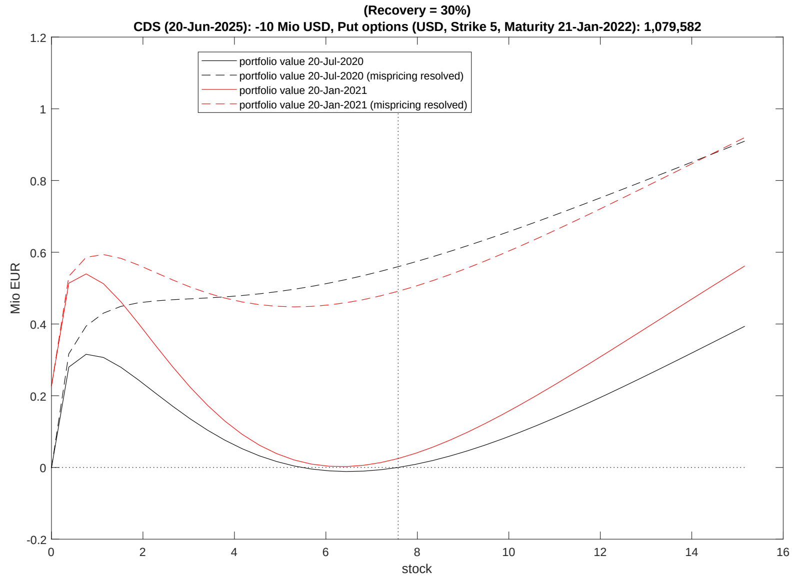

Fig. 2: The sensitivity of the described position when only one put option (with strike 5) is used for hedging.

Figures 2, 3 and 4 depict the sensitivities of the hedged position using three different possible hedges, each of which seeking to hedge both jump-to-default and delta risk. Figure 2 uses only one put option H to hedge the exposure of L. We know from Figure 1 that the strike level 5 implies the delta-to-jump ratio $-\Delta H/\Phi H$ that is closest to the delta-to-jump ratio of L. Consequently, this strike is opted for, and the number of options is chosen such that the jump-to-default risk is eliminated. As theory suggests, the delta risk is also almost eliminated, although not perfectly due to the fact that the delta-to-jump ratio of the put is slightly smaller than that of L, see Figure 1.
Figure 3 uses two put options with different strikes for hedging L. According to Figure 1 one strike level must be chosen from the set $\{3, 5\}$ and the other from the set $\{8, 10, 12\}$ in order for (2) to be satisfied. We opted for the strike pair $(3, 10)$, and Figure 3 illustrates the associated sensitivities. Apparently, jump-to-default risk and delta risk are both eliminated completely, since the numbers of the put options are chosen according to our unique solution presented.
Finally, Figure 4 illustrates the situation when H_1 is a long CDS with maturity in June 2021.

References

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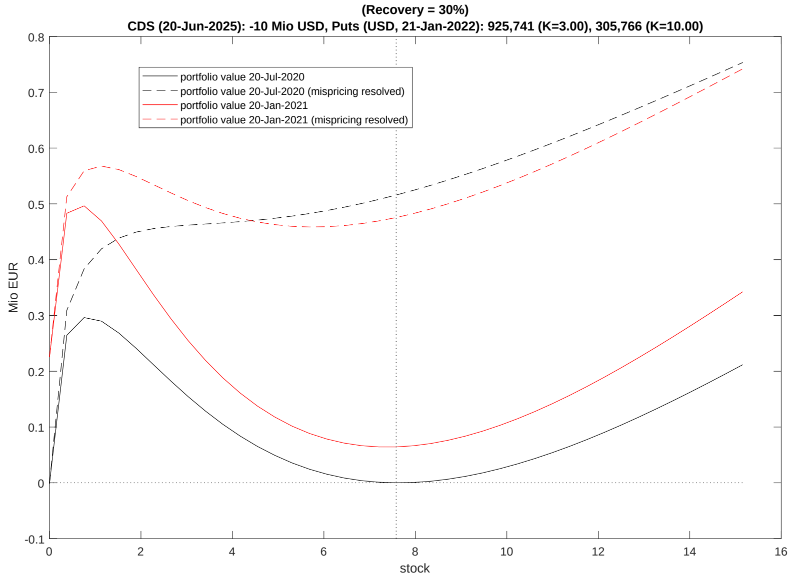

Fig. 3: The sensitivity of the described position when two put options (with strikes 3 and 10) are used for hedging.

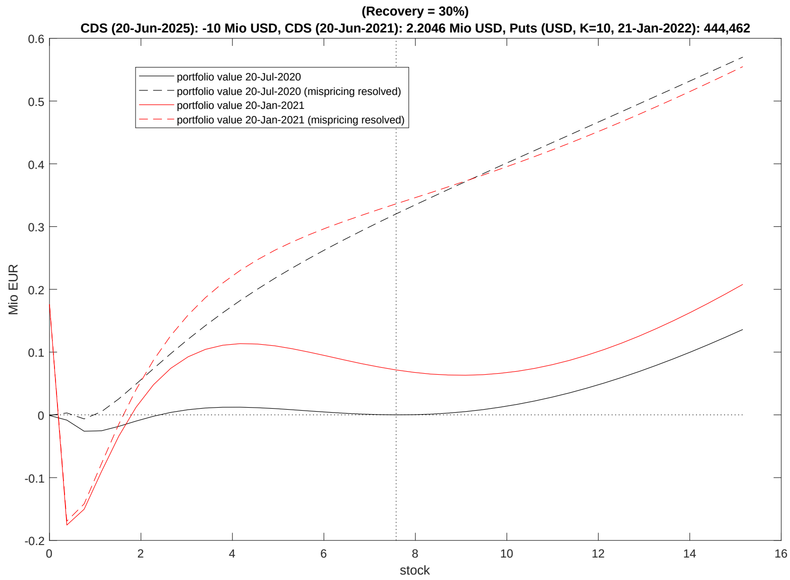

Fig. 4: The sensitivity of the described position when one put option (with strike 10) and a shorter-dated long CDS are used for hedging.

Appendix: Delta-to-jump ratio of European put option in the Black-Scholes model

Consider the Black-Scholes model with interest rate $r \in \mathbb{R}$, volatility $\sigma > 0$, and maturity $T > 0$. For a European put option with strike K the delta-to-jump ratio is given by

$$-\frac{\Delta P}{K-P} = \frac{\mathcal{N}\big(-d_1(K)\big)}{K\left\{1-e^{-rT}\mathcal{N}\big(-d_2(K)\big)\right\}+S_0\,\mathcal{N}\big(-d_1(K)\big)},$$

where $\mathcal{N}$ denotes the cdf of a standard normally distributed random variable and

$$d_1(K) = \frac{\log(S_0/K) + (r + \sigma^2/2)\,T}{\sigma\,\sqrt{T}},$$
$$d_2(K) = \frac{\log(S_0/K) + (r - \sigma^2/2)\,T}{\sigma\,\sqrt{T}}.$$

Taking the derivative with respect to the strike K, we observe with a tedious computation that

$$\begin{aligned}
&\frac{\partial}{\partial K}\Big(-\frac{\Delta P}{K-P}\Big) \geq 0 \\
&\Leftrightarrow \mathcal{N}'\big(-d_2(K)\big)\,\mathcal{N}\big(-d_1(K)\big)\,e^{-r\,T} + \big\{1 - e^{-r\,T}\,\mathcal{N}\big(-d_2(K)\big)\big\}\times \\
&\qquad \times \big[\mathcal{N}'\big(-d_1(K)\big) - \sigma\,\sqrt{T}\,\mathcal{N}\big(-d_1(K)\big)\big] \geq 0 \\
&\Leftrightarrow e^{-r\,T}\,\Big(\mathcal{N}'(-d_2)\,\mathcal{N}(-d_1) - \mathcal{N}'(-d_1)\,\mathcal{N}(-d_2)\Big) \\
&\qquad + \sigma\,\sqrt{T}\,\mathcal{N}(-d_1)\,\Big(e^{-r\,T}\,\mathcal{N}(-d_2) - 1\Big) + \mathcal{N}'(-d_1) \geq 0,
\end{aligned} \tag{4}$$

where we have dropped the dependence of d_i on K for the sake of a more compact notation in the last equivalence, $i = 1, 2$. (Boyd, Vandenberghe, 2004, Exercise 3.54, p. 123) shows that $\mathcal{N}$ is log-concave, meaning that

$$\frac{\mathcal{N}'(x)}{\mathcal{N}(x)} \geq \frac{\mathcal{N}''(x)}{\mathcal{N}'(x)} = -x, \quad x \in \mathbb{R}. \tag{5}$$

From this, we observe for $x \leq 0$ that $\mathcal{N}'(x)/\mathcal{N}(x) \geq -x \geq 0$. Since $d_1(K) \geq d_2(K)$, under the assumption that $d_2(K) \geq 0$ this implies that the first term in (4) is non-negative. Notice that $d_2(K) \geq 0$ means that

$$K \leq S_0\,e^{\left(r - \frac{\sigma^2}{2}\right)\,T}.$$

In this case, with the very same estimate $\mathcal{N}'(-d_1) \geq \mathcal{N}(-d_1)\,d_1$ we observe

$$\begin{aligned}
&\sigma\,\sqrt{T}\,\mathcal{N}(-d_1)\,\Big(e^{-r\,T}\,\mathcal{N}(-d_2) - 1\Big) + \mathcal{N}'(-d_1) \\
&\quad \geq \mathcal{N}(-d_1)\,\Big(d_1 - \sigma\,\sqrt{T}\big(1 - \exp(-r\,T)\,\mathcal{N}(-d_2)\big)\Big) \\
&\quad \geq \mathcal{N}(-d_1)\,\big(d_2 + \exp(-r\,T)\,\mathcal{N}(-d_2)\big) \geq 0,
\end{aligned}$$

where the penultimate inequality uses $d_2 + \sigma\,\sqrt{T} = d_1$ and the last estimate relies on our assumption $d_2 \geq 0$. Ultimately, we have thus shown the following lemma.

Lemma 3.1 (Delta-to-jump for OTM puts)
The delta-to-jump ratio of a European put option in the Black–Scholes model is increasing in the strike price K for strike prices satisfying

$$K \leq S_0\,e^{\left(r - \frac{\sigma^2}{2}\right)\,T}.$$

Finally, we find it important to point out that the delta-to-jump ratio of a put option in general needs not be increasing in the strike price. To this end, Figure 5 depicts the delta-to-jump ratio of a European put in a particular choice of parameters (large r, σ, T), for which the ratio begins to decrease for large strikes. Lemma 3.1 only guarantees increasingness for strikes below the red dotted line.

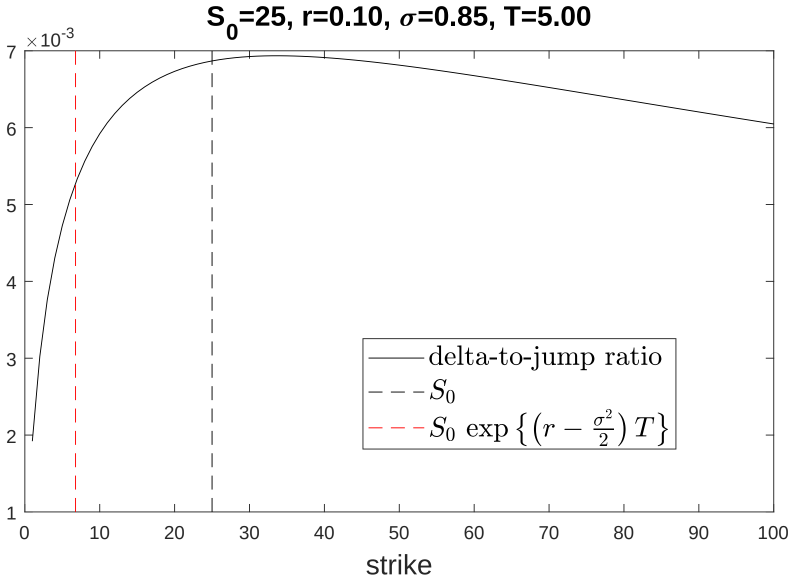

Fig. 5: Delta-to-jump ratio of a European put in a particular Black–Scholes model, whose parameters are such that increasingness in the strike is only true for small strikes.

Generally speaking, if the maturity T is chosen very large, and $\sigma^2/2 > r > 0$, then one can construct examples for which the red dotted line is very close to zero and at the same time the delta-to-jump ratio is decreasing in K for almost all strikes. However, in practice one typically has $T < 2$ and r much smaller than 10% as depicted in Figure 5, so that increasingness of the delta-to-jump ratio for OTM strikes is the “typical” case.