



**DIFFERENTIAL DISCOUNTING
FOR COLLATERALIZED CDS:
WHY, HOW, AND WHAT ARE
THE UNDERLYING
ASSUMPTIONS?**

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Abstract In the post-crisis derivative pricing literature one often finds the statement that the pricing of a collateralized derivative is the same as the pricing of its uncollateralized counterpart, only with adjusted discount factors. We re-derive this statement in the particular case of a CDS by a cashflow-oriented approach, emphasizing the hidden assumptions underlying this derivation. In particular, said pricing technique completely ignores counterparty credit risk and – if more than one currency is involved – necessarily relies on the assumption of funding-dependent discounting. If collateral is posted, the interest rate that has to be paid on the collateral dictates the choice of discounting curve.

1 Introduction A credit default swap (CDS) is an insurance contract between two counterparties. One party, the protection buyer, pays an upfront amount and a periodic insurance premium, called the CDS running coupon, to the other party, the protection seller. In return, the protection seller has to compensate the protection buyer for losses occurring from a credit event of the underlying reference entity. A credit event is a notion that is precisely defined in the CDS contract, comprising several structural changes in the reference entity's creditworthiness, such as a failure to pay a due coupon or a restructuring of its capital structure. During the lifetime of the contract, the market's opinion on the creditworthiness of the reference entity changes all the time, which can be observed by daily price fluctuations of CDS upfront payments. Regulatory rules demand that both counterparties mark-to-market the CDS on a daily basis in order to take these fluctuations into account. This means that at each time point one party is in-the-money and the other party is out-of-the-money. In order to minimize counterparty credit risk it has become market standard that the party which is out-of-the-money posts collateral to the other party. The amount of the collateral is typically determined by the current market price of the CDS, on which therefore both parties must agree. The party receiving the collateral has to pay interest on this collateral at a rate customary in the market. This is typically an overnight rate determined by a standardized index. Moreover, the collateral need not be posted in the same currency as the one in which the CDS is denominated. For instance, it is not unusual that in a USD-denominated CDS the party which is in-the-money receives the collateral in EUR, and then has to pay EONIA on this collateral to the counterparty. If the party out-of-the-money is a USD-based bank, this might be favorable compared with posting the collateral in USD and re-



ceiving Fed Funds rate on it, because the cross currency basis spread between EONIA and Fed Funds rate can be earned. For instance, on August 16, 2008, a one-year cross currency swap exchanges three-month tenor EURIBOR minus a spread of 38 bps against three-month tenor USD LIBOR. The collateral currency is agreed upon between both counterparties in a so-called credit support annex (CSA). Often such CSA agreements are made between two counterparties globally for all their outstanding derivative contracts, explaining why currency “mismatches” between CDS cash flows and collateral cash flows are not uncommon. As a consequence of such a CSA agreement the CDS might bear an implicit funding advantage for one of both counterparties at inception. And this implicit optionality has to be taken care of in the CDS valuation.

It has already been highlighted in the literature that implicit funding advantages arising from CSAs affect the pricing of the respective derivatives. For instance, Johannes, Sundaresan (2007) treat the pricing of collateralized interest rate swaps and Piterbarg (2010), whose derivation has been corrected in Brigo et al. (2012a), considers the pricing of collateralized stock derivatives. Based on a hedging argument, the results of Piterbarg (2010) imply that the pricing of collateralized stock derivatives equals the pricing of uncollateralized stock derivatives, only the discounting curve has to be adjusted appropriately. A similar result, although derived differently, is provided also in Hull, White (2012). A much more general setup including all types of derivatives and imposing minimal restrictions on the collateral specifications is considered in Pallavicini et al. (2011), a reference which provides a nice survey of further references in its introduction. In a similar spirit, Fries (2010) also considers quite basic derivatives but under quite general axioms regarding counterparty risk and funding issues, highlighting that the cost of funding topic is far from trivial. Regarding the pricing of collateralized derivatives when more than one currency is involved, Fujii et al. (2010) consider implications for generic derivatives and curve construction, while Castagna (2012) treats the pricing of stock derivatives in a purely Brownian setup.

The research note Doctor et al. (2012) from August 2012 by JP Morgan argues that the discounting curve when valuing CDS with collateral posted in a foreign currency should be adjusted by the cross-currency basis spread. However, neither a rigorous mathematical derivation nor a reference to the respective literature¹ is provided in that note. The main purpose of the present article is (a) to provide such a derivation and provide the respective references to the academic literature, and (b) to explain in an easy manner why (and under which assumptions) collateralization affects CDS pricing only via an adjustment of the discount factors. On the first glimpse this appears to be an ad hoc adjustment because no cash flows from the collateral account are considered in the pricing at all.

We choose the simplest possible setup required in order to cope with the tasks (a) and (b). The mathematical innovation of the present note is minimal, we basically re-derive the result of Fujii

¹Such as, e.g., to (Fujii et al., 2010, Theorem 1).



et al. (2010) for the particular case of a CDS, but we additionally discuss in detail the implications of funding-dependent discounting on pricing, a topic which recently triggered a vivid debate in the marketplace, see, e.g., Carver (2012). Our setup comprises the following convenient assumptions, allowing us to solely focus on the effects of the CSA on the pricing:

- Independence of the information regarding the reference entity's credit risk and the interest rate evolution.
- A two-way, zero-threshold collateral agreement with cash collateral being transferred continuously.

As noted by Hull, White (2012), under the second assumption the collateral is a perfect hedge for losses due to default so that credit value adjustments (CVA) and debt value adjustments (DVA) are both zero. Therefore, we may completely ignore counterparty credit risk assuming that neither protection buyer nor protection seller can default. Clearly, a relaxation of these assumptions is desirable and important, and there already exists very good literature on the effects of counterparty risk and/or collateralization for CDS valuation, see, e.g. the quite recommendable references Brigo, Capponi (2010); Crépey et al. (2010); Brigo et al. (2012b), who consider a much more general setup, including funding issues in particular. However, we think that a simplified setup, such as ours above or the one applied in Fujii et al. (2010), is still of interest because of the following reasons:

- The narrow focus allows us to concentrate specifically on the cross-currency effects and the practical implications thereof. In particular, we illustrate that in a two currency setup it is important to work with interest rate bootstraps that are adjusted by cross-currency basis spreads.
- A more general setup leads to pricing formulas that differ significantly from market standard pricing routines. Their incorporation into industrial software systems requires huge efforts, so that on the short run the market is in need for simpler ad hoc adjustments accounting for collateralization in an appropriate manner.
- When the collateral amount is computed in practice, both counterparties must agree on a (preferably simple) common evaluation methodology. The reference Doctor et al. (2012) indicates that the market will move to a standard market pricing mechanism based on differential discounting, i.e. applying the traditional (uncollateralized) pricing routine, only with adjusted discount factors. Therefore, we find it important to provide a practically oriented audience with a cashflow-oriented, rigorous mathematical explanation of how this methodology can be justified.

Again, as a final remark we would like to make people aware that treating collateralization by simply changing the discount factors relies heavily on the simplifying assumptions above – in particular completely ignores counterparty credit risk issues. Therefore we urge readers to read the more theoretical article Brigo et al. (2012b) for implications when these assumptions are relaxed.

2 Technical part

2.1 Notation We consider a CDS in currency y , but the collateral for the CDS is posted in currency x . In order to be general we allow for $x = y$, but of course the case $x \neq y$ is more interesting. Without loss of generality we evaluate the CDS from the protection buyer's point of view in units of y . The CDS maturity is denoted by T . According to ISDA standards, CDS premium payments are typically made quarterly. However, we assume that the CDS running coupon, as well as all other interest payments, are paid continuously. This makes the notations of the present paper a lot easier, since we can replace sums by integrals. The discrete counterparts of the formulas we derive can be obtained by discretizing the respective integrals later on. For example, consider the present value of a floating bond with interest payment dates $0 =: t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} := T$. At each time point $t_i^{(n)}$ the bond pays the forward rate $f(t_{i-1}^{(n)}, t_i^{(n)})$ which was fixed at $t_{i-1}^{(n)}$ for the period $[t_{i-1}^{(n)}, t_i^{(n)}]$. The discount factor for payments at time t is denoted by $DF(t)$. When the partition of the interval $[0, T]$ is made finer and finer, it holds true that

$$\sum_{i=1}^n f(t_{i-1}^{(n)}, t_i^{(n)}) (t_i^{(n)} - t_{i-1}^{(n)}) DF(t_i^{(n)}) \longrightarrow \int_0^T r(t) DF(t) dt, \quad n \rightarrow \infty,$$

where $r(t) := \lim_{s \downarrow t} f(t, s)$ denotes the instantaneous short rate associated with the forward rate surface $\{f(t, s)\}_{s \geq t \geq 0}$. Clearly, this shows that the integral expression is much more compact and less messy than the sum expression, which is the sole reason for working in continuous time. Moreover, we introduce the following notations:

- All involved stochastic objects are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where \mathcal{F}_t contains all information available in the market at time t .
- τ denotes the default time of the reference entity of the CDS. We assume that it has the default intensity $\lambda = \{\lambda(t)\}$, which is adapted to $\{\mathcal{F}_t\}$. This means that the conditional survival function and density, conditioned on the event $\{\tau > t\}$ and the available information \mathcal{F}_t at time t , are given by

$$\begin{aligned} \bar{F}_{\tau|\mathcal{F}_t}(u) &:= \mathbb{P}(\tau > u | \mathcal{F}_t) = \mathbb{E} \left[e^{-\int_t^u \lambda(v) dv} \mid \mathcal{F}_t \right], \quad u \geq t \\ f_{\tau|\mathcal{F}_t}(u) &:= -\frac{d}{du} \bar{F}_{\tau|\mathcal{F}_t}(u) = \mathbb{E} \left[\lambda(u) e^{-\int_t^u \lambda(v) dv} \mid \mathcal{F}_t \right], \quad u \geq t. \end{aligned}$$

Intuitively, $f_{\tau|\mathcal{F}_t}$ is the density of τ as the market observes it at time t , and $\bar{F}_{\tau|\mathcal{F}_t}(u)$ is the survival probability until time u as the market sees it at time t .

- $R \in [0, 1]$ denotes the constant recovery rate underlying all CDS valuations, s the CDS running coupon, upf the CDS upfront payment.



- $r^C = \{r^C(t)\}_{t \geq 0}$ denotes the continuous short rate that has to be paid on the collateral, which is typically an overnight rate such as EONIA (if $x = EUR$) or Fed Funds rate (if $x = USD$). The rates $r^y = \{r^y(t)\}_{t \geq 0}$, respectively $r^x = \{r^x(t)\}_{t \geq 0}$ are the ones which the protection buyer uses for discounting cash flows in the respective currencies. As mentioned earlier, we impose the assumption that the interest rate evolution (r^x, r^y, r^C) is independent of the credit information evolution λ .
- $FX^{x/y}(t)$ is the x -value for one unit of currency y at time t , and $FX^{y/x}(t)$ its reciprocal.
- We denote by C_t the collateral posted by the protection seller to the protection buyer at time t in units of x , so that $FX^{y/x}(t) C_t$ denotes the collateral amount at time t in units of y . Note that this number can be negative in case the protection seller is in-the-money.
- We denote by V_t the market y -value of the CDS at time t , which is the quantity we are interested in.

2.2 A primer on funding-dependent vs. risk-free discounting

Fujii et al. (2010) and Hull, White (2012) interpret the discounting rates (r^x, r^y) as risk-free rates – even though other authors, such as Piterbarg (2010), interpret these rates as funding rates. However, as Piterbarg (2010) mentions, in reality these funding rates depend on the creditworthiness of the protection buyer, which stands in glaring contrast to the interpretation of (r^x, r^y) being risk-free; and it is inconsistent with the assumption of the protection buyer being free of default risk. If one decides to include the possibility of a default of the protection buyer, the whole picture becomes dramatically more difficult, which is why this effect is often ignored (also in the present note). For instance, in such a case it is only reasonable to assume that the protection buyer's funding rate is different from the rate at which he can invest cash in a risk-free manner. Think of a fund manager who has to promise his investors a certain target rate (after administration costs) in order to raise money, i.e. this target rate might be considered as funding rate. However, the same investor need not be able to earn the target rate with a risk-free strategy. But in order to remain in business he certainly has to make sure that he earns the target rate on average. Mathematically, any future cash flow can be zero in case the protection buyer defaults, and in order to compute the cash flow's expected present value one has to discount it not only at the risk-free rate but additionally multiply it by the protection buyer's survival probability. Interpreting the sum of risk-free rate and the protection buyer's hazard rate as his funding rate², we end up with the viewpoint of Piterbarg (2010) but have to keep in mind that (r^x, r^y) are no longer rates that can be earned with a risk-free trading strategy. Rather they have to be interpreted as target rates, relative to which each cash flow is evaluated, like a numeraire.

In general, any market participant evaluates the price of a derivative in her book using a certain model. Hence, the price of the de-

²Under zero recovery assumption.



derivative on the market participant's book is a function of some discounting curve and other model parameters (e.g., such as default probabilities or volatility parameters etc.). Referring to nomenclature of classical arbitrage pricing theory, the applied discounting curve typically corresponds to the choice of a numeraire \mathcal{N} and the remaining model parameters determine the pricing measure $\mathbb{Q}^{\mathcal{N}}$. When two counterparties trade a derivative, they agree on a price, namely the market price for the derivative³. However, if both counterparties evaluate the same derivative using different, idiosyncratic discounting curves, e.g. according to their respective funding rates, then both have different numeraires. In order for both to mark the derivative to the given market price, they have to adjust their pricing measures $\mathbb{Q}^{\mathcal{N}}$ accordingly. This means that the measure $\mathbb{Q}^{\mathcal{N}}$ is subjective, since it depends on the choice of numeraire \mathcal{N} . Note, however, that the price is not subjective, because it has been agreed upon by both parties at inception of the derivative, and is agreed upon on a daily basis between both parties, because collateral is exchanged between them on a daily basis. Assume for instance that one counterparty is a major investment bank and uses OIS-based discounting, whereas the other counterparty is a hedge fund ("she") using her target rate OIS+300 bps for discounting, because the latter rate is what she promises to earn for her investors. If both parties enter a CDS contract, this implies that the hedge fund's risk-neutral default probabilities for the reference entity are significantly lower compared with the bank's risk-neutral default probabilities. Such a pricing technique clearly renders the meaning of market-implied survival probabilities subjective. In particular, market-wide use of such funding-dependent pricing renders statements like "the market-implied default probabilities for Spain are ...", which can frequently be read in the papers, meaningless to some extent. In order for the "market-implied" default probabilities to be meaningfully determined, there must be something like a "market-standard" numeraire \mathcal{M} which the statement implicitly is based on. Indeed, one can argue that there exists a certain market standard discounting curve, in the post-crisis market environment typically being OIS-based discounting. Then it is arguable whether marking-to-market means fitting the price with one's idiosyncratic numeraire \mathcal{N} and (then necessarily) subjective pricing measure $\mathbb{Q}^{\mathcal{N}}$, or whether it means fitting the market price with the market standard numeraire \mathcal{M} and respective market-implied pricing measure $\mathbb{Q}^{\mathcal{M}}$. But definitely wrong would be to use one's subjective discounting curve with market-implied pricing measure, because this leads to subjective prices violating the law of one price.

Unfortunately, even if one decides against funding-dependent pricing and in favor of market-standard discounting, when more than one currency is involved the picture becomes puzzling. This is because in the post-crisis market environment the cross-currency basis spreads have widened significantly. If one decides for OIS-based discounting in one's home currency, as is current market standard, then arbitrage arguments imply that foreign currency discount factors have to be bootstrapped using cross-

³+/- bid-offer spread, which we ignore within our model cosmos.

currency basis spread adjustments, as described in Fujii et al. (2010). But this approach already implies that a European investor uses another numeraire than an American investor. For instance, if they enter an interest rate swap, the pricing models of both parties necessarily imply different dynamics for the underlying forward rates, i.e. they work with different pricing measures \mathbb{Q}^{EUR} , resp. \mathbb{Q}^{USD} , although being not as evidently different as in the aforementioned CDS example, but the difference being more subtle. Long story short, we conclude this brief discussion with two facts to be remembered:

- The choice of discounting curve is a choice of numeraire but must not violate the law of one price.
- In the current market environment, no one can get around funding-dependent discounting, unless he is acting in a single currency only.

2.3 Pricing derivation

Let us now consider a time point t during the lifetime of the CDS. The protection buyer holds the collateral amount C_t . If $C_t > 0$, he can invest the collateral amount at rate r^x , and has to pay interest r^C on the collateral to the protection buyer. If $C_t < 0$, he misses the former investment opportunity and hence pays an opportunity cost of capital, but therefore receives the interest r^C from the protection seller. In the sequel, we gather precisely the profit and loss components stemming from the collateral account, implicitly assuming that $C_t > 0$ in our wording. The case $C_t < 0$ is also included mathematically, only our wording takes the perspective of the protection buyer being in-the-money.

- (1) The (random) present x -value of the protection buyer's earnings (inflows⁴) I_t^x at time t from managing the collateral account, conditioned on the event $\{\tau > t\}$, is given by

$$I_t^x := C_t + \int_t^T \left(\frac{d}{du} C_u \right) \mathbf{1}_{\{\tau > u\}} e^{-\int_t^u r^x(v) dv} du.$$

The integral expression intuitively means that the future in- and outflows on the collateral account due to mark-to market changes, given the reference entity is still alive, are accounted for by discounting them at the rate r^x into time t . Using the discounting rate r^x implicitly means that we assume the collateral account is invested at the rate r^x (re-hypothecation⁵). Notice in particular that we use a heuristical argument here by assuming that the function $t \mapsto C_t$ is almost surely differentiable until default. However, it is shown later that this assumption is indeed justified, since Remark 2.2 below will imply differentiability of $t \mapsto C_t$. The appearance of the derivative becomes intuitively clear when discretizing the integral to a sum, yielding

$$\sum_{i=1}^n (C_{t_i} - C_{t_{i-1}}) \mathbf{1}_{\{\tau > t_{i-1}\}} e^{-\int_t^{t_{i-1}} r^x(v) dv},$$

⁴These are only inflows depending on the sign, so the wording is slightly misleading.

⁵Sometimes, re-hypothecation is prohibited, in which case the picture becomes more difficult.

which simply means that the daily change in the collateral has to be paid to the protection seller, i.e. if $C_{t_{i-1}} < C_{t_i}$ the protection buyer has an outflow, and vice versa. Applying partial integration on the integral expression and $C_T = 0$, we end up with

$$I_t^x = \int_t^T C_u 1_{\{\tau > u\}} e^{-\int_t^u r^x(v) dv} r^x(u) du + 1_{\{\tau < T\}} e^{-\int_t^\tau r^x(v) dv} C_\tau.$$

This is an intuitive formula, showing that the rate r^x is earned on the collateral until default, in which case the protection buyer ends up with the collateral C_τ at default. Since we want to evaluate the CDS in units of y , we have to convert this discounted cash flow stream into y -units, yielding

$$I_t^y = \int_t^T C_u FX^{y/x}(u) 1_{\{\tau > u\}} e^{-\int_t^u r^y(v) dv} r^x(u) du + 1_{\{\tau < T\}} e^{-\int_t^\tau r^y(v) dv} C_\tau FX^{y/x}(\tau).$$

Explaining the first summand, the value of the cash flow $C_u r^x(u)$ at time u in y -units is $C_u FX^{y/x}(u) r^x(u)$, and a y -cash flow at time u has to be discounted at the rate r^y . The second summand is explained analogously.

- (2) The (random) present x -value of the protection buyer's costs (outflows) O_t^x on the collateral account at time t , conditioned on the event $\{\tau > t\}$, is given by

$$O_t^x = \int_t^T C_u 1_{\{\tau > u\}} e^{-\int_t^u r^x(v) dv} r^C(u) du + 1_{\{\tau < T\}} e^{-\int_t^\tau r^x(v) dv} C_\tau.$$

In case the reference entity defaults, the CDS contract ends and the outstanding collateral is repaid to the protection seller, explaining the occurrence of the second term. Again, conversion into units of y yields

$$O_t^y = \int_t^T C_u FX^{y/x}(u) 1_{\{\tau > u\}} e^{-\int_t^u r^y(v) dv} r^C(u) du + 1_{\{\tau < T\}} e^{-\int_t^\tau r^y(v) dv} C_\tau FX^{y/x}(\tau).$$

In total, this gives the present y -value of managing the collateral account at time t as inflows minus outflows, i.e. $I_t^y - O_t^y =$

$$\int_t^T C_u FX^{y/x}(u) 1_{\{\tau > u\}} e^{-\int_t^u r^y(v) dv} (r^x(u) - r^C(u)) du.$$

Again, this is an intuitive formula, showing that the rate r^x is earned on the collateral while the rate r^C has to be paid on the collateral until default.

The market value of the CDS at time t consists of two accounts: on the one hand, one has to account for all the future cash flows generated by the CDS contract, i.e. premium payments to be made by the protection buyer and default compensation payments



to be made by the protection seller. On the other hand, the collateral account has to be considered as well, because we have just seen that there might be an opportunity cost of capital due to the fact that collateral is posted and $r^x \neq r^C$. Summing up, we end up with the following equation for the market y -value V_t of the CDS contract, conditioned on the event $\{\tau > t\}$ and the available information \mathcal{F}_t at time t :

$$\begin{aligned}
 V_t = 1_{\{\tau > t\}} \mathbb{E} & \left[(1 - R) e^{-\int_t^\tau r^y(u) du} 1_{\{\tau \leq T\}} \right. \\
 & \left. - s \int_t^T 1_{\{\tau > u\}} e^{-\int_t^u r^y(v) dv} du \middle| \mathcal{F}_t \right] \\
 + 1_{\{\tau > t\}} \mathbb{E} & \left[\int_t^T 1_{\{\tau > u\}} C_u F X^{y/x}(u) \times \right. \\
 & \left. \times (r^x(u) - r^C(u)) e^{-\int_t^u r^y(v) dv} du \middle| \mathcal{F}_t \right]. \quad (1)
 \end{aligned}$$

In particular, if the CDS is not collateralized, or if $r^x = r^C$, the second term vanishes and V_t is given by the usual formula. Moreover, Formula (1) is consistent with findings in Piterbarg (2010), who derives the market values for stock derivatives under CSA in a Brownian setup, compare Formula (3) in that paper.

Generally speaking, Formula (1) is not of direct use for CDS valuation because the collateral process $\{C_t\}$ is a priori unknown. However, it is reasonable to assume the following axiom:

- (A) $C_t F X^{y/x}(t) = V_t$ for all $t \in [0, T]$
 ("two-way, zero-threshold, continuous CSA")

Both counterparties in the CDS contract have to agree on the current collateral amount C_t on a daily basis. It is only natural to assume that they agree on the collateral amount being equal to the actual market value of the CDS, hence axiom (A). Under this axiom Formula (1) still is not of direct use for pricing the collateralized CDS contract, because it now manifests an equality the process $\{V_t\}$ has to satisfy:

$$\begin{aligned}
 V_t = 1_{\{\tau > t\}} \mathbb{E} & \left[(1 - R) e^{-\int_t^\tau r^y(u) du} 1_{\{\tau \leq T\}} \right. \\
 & \left. - s \int_t^T 1_{\{\tau > u\}} e^{-\int_t^u r^y(v) dv} du \middle| \mathcal{F}_t \right] \\
 + 1_{\{\tau > t\}} \mathbb{E} & \left[\int_t^T 1_{\{\tau > u\}} V_u \times \right. \\
 & \left. \times (r^x(u) - r^C(u)) e^{-\int_t^u r^y(v) dv} du \middle| \mathcal{F}_t \right]. \quad (2)
 \end{aligned}$$

However, the following lemma provides a solution to this equation, yielding a useful pricing formula for the collateralized CDS contract.

Lemma 2.1 (Pricing formula for collateralized CDS)

A solution to equation (2) is given by the stochastic process $\{V_t\}_{t \in T}$, defined as

$$\begin{aligned}
 V_t = 1_{\{\tau > t\}} \mathbb{E} & \left[(1 - R) e^{-\int_t^\tau r^y(u) - (r^x(u) - r^C(u)) du} 1_{\{\tau \leq T\}} \right. \\
 & \left. - s \int_t^T 1_{\{\tau > u\}} e^{-\int_t^u r^y(v) - (r^x(v) - r^C(v)) dv} du \middle| \mathcal{F}_t \right]. \quad (3)
 \end{aligned}$$

Remark 2.2

Using independence between λ and (r^C, r^x, r^y) , Formula (3) might as well be stated as

$$V_t = 1_{\{\tau > t\}} \int_t^T \left[(1 - R) f_{\tau|\mathcal{F}_t}(u) - s \bar{F}_{\tau|\mathcal{F}_t}(u) \right] \times \\ \times \mathbb{E} \left[e^{-\int_t^u r^y(v) - (r^x(v) - r^C(v)) dv} \middle| \mathcal{F}_t \right] du.$$

Proof (of Lemma 2.1)

Plugging in the claimed candidate into (2) verifies the lemma, since on the event $\{\tau > t\}$ we have

$$\begin{aligned} & \mathbb{E} \left[\int_t^T 1_{\{\tau > u\}} V_u (r^x(u) - r^C(u)) e^{-\int_t^u r^y(v) dv} du \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\int_t^T 1_{\{\tau > u\}} \mathbb{E} \left[(1 - R) e^{-\int_u^{\tau} r^y(v) - (r^x(v) - r^C(v)) dv} 1_{\{\tau \leq T\}} \right. \right. \\ & \quad \left. \left. - s \int_u^T 1_{\{\tau > v\}} e^{-\int_u^v r^y(m) - (r^x(m) - r^C(m)) dm} dv \middle| \mathcal{F}_u \right] \times \right. \\ & \quad \left. \times (r^x(u) - r^C(u)) e^{-\int_t^u r^y(v) dv} du \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\int_t^T \int_u^T \left[(1 - R) f_{\tau|\mathcal{F}_u}(v) - s \bar{F}_{\tau|\mathcal{F}_u}(v) \right] \times \right. \\ & \quad \left. \times e^{-\int_u^v r^y(m) - (r^x(m) - r^C(m)) dm} dv \times \right. \\ & \quad \left. \times (r^x(u) - r^C(u)) e^{-\int_t^u r^y(m) dm} du \middle| \mathcal{F}_t \right] \\ &\stackrel{(*)}{=} \int_t^T \int_u^T \left[(1 - R) f_{\tau|\mathcal{F}_t}(v) - s \bar{F}_{\tau|\mathcal{F}_t}(v) \right] \times \\ & \quad \times \mathbb{E} \left[e^{-\int_u^v r^y(m) - (r^x(m) - r^C(m)) dm} \times \right. \\ & \quad \left. \times (r^x(u) - r^C(u)) e^{-\int_t^u r^y(m) dm} \middle| \mathcal{F}_t \right] dv du \\ &\stackrel{(**)}{=} \int_t^T \left[(1 - R) f_{\tau|\mathcal{F}_t}(v) - s \bar{F}_{\tau|\mathcal{F}_t}(v) \right] \times \\ & \quad \times \mathbb{E} \left[\int_t^v e^{-\int_t^v r^y(m) - (r^x(m) - r^C(m)) dm} \times \right. \\ & \quad \left. \times (r^x(u) - r^C(u)) e^{-\int_t^u r^y(m) dm} du \middle| \mathcal{F}_t \right] dv \\ &= \int_t^T \left[(1 - R) f_{\tau|\mathcal{F}_t}(v) - s \bar{F}_{\tau|\mathcal{F}_t}(v) \right] \times \\ & \quad \times \mathbb{E} \left[e^{-\int_t^v r^y(m) - (r^x(m) - r^C(m)) dm} - e^{-\int_t^v r^y(m) dm} \middle| \mathcal{F}_t \right] dv \\ &= V_t - \int_t^T \left[(1 - R) f_{\tau|\mathcal{F}_t}(v) - s \bar{F}_{\tau|\mathcal{F}_t}(v) \right] \times \\ & \quad \times \mathbb{E} \left[e^{-\int_t^v r^y(m) dm} \middle| \mathcal{F}_t \right] dv. \end{aligned}$$

Equality (*) above requires independence between the hazard rate λ and the interest rate evolution (r^x, r^y, r^C) , and Equality (**) changes the order of integration. \square

2.4 Commenting on the derived formula

We collect some remarks that should be made on the derived formula.

(a) Lemma 2.1 shows that the value of the collateralized CDS agrees with the value of the uncollateralized CDS, only with



another discounting curve, namely $r^y - (r^x - r^C)$, which is precisely the result of (Fujii et al., 2010, Theorem 1). If $x = y$, the interest rate r^C to be paid on the collateral dictates the choice of discounting curve, which is in line with results of Piterbarg (2010) in the context of stock derivatives, derived in a Brownian setup via the Feynman-Kac theorem, a corrected derivation is given in Brigo et al. (2012a). It also is in line with the argumentation of Hull, White (2012), as well as with the result of (Pallavicini et al., 2011, p. 12) in the special case of perfect collateralization with symmetric CSA.

- (b) If $r^C = r^x$, i.e. if the rate to be paid on the collateral equals the x -discounting rate of the protection buyer, then the collateral account can be ignored completely.
- (c) In the single-currency case, the resulting pricing formula is completely independent of the discounting curve $r^x = r^y$, hence the choice of numeraire does not even matter! However, in the two-currency case the discounting rate $r^y - (r^x - r^C) = (r^y - r^x) + r^C$ splits into two parts: r^C does not depend on the choice of discounting rates, but $r^y - r^x$ depends on this choice, because the cross-currency funding needs to be incorporated into the pricing.
- (d) Remark 2.2 shows that $t \mapsto V_t$ is indeed almost surely differentiable until default, justifying our earlier assumption.
- (e) The CDS upfront payment upf of the CDS is determined as $upf = V_0$.

Lemma 2.1 implies a mathematically rigorous derivation of a proposal made by Doctor et al. (2012), namely that the collateralized CDS should be evaluated like in the traditional approach, only the discounting curve needs to be adjusted by the cross-currency basis swap spread. As pointed out in that reference, this is of particular interest if the currencies x and y are different. Suppose that the CDS is denominated in $y = USD$ but collateral is posted in currency $x = EUR$, which is the example considered in Doctor et al. (2012). It is market standard that the rate r^C to be paid on the collateral is based on some index of EUR -denominated overnight rates, which typically is the EONIA index. If the protection buyer is a USD investor it might be reasonable to assume that his USD -discounting rate r^y is derived from some market quoted USD -denominated interest rate swap (or overnight) rates. However, since he has to pay the EUR -denominated EONIA rate r^C on the collateral, his EUR -discounting rate r^x and his USD -discounting rate r^y might not be the same but actually differ by means of the cross-currency basis spread between both currencies. Since the beginning of the credit crisis, when the international money transfer ran dry, an extensive increase in this spread has been observed. Therefore, using Formula (3) for CDS valuation, Doctor et al. (2012) propose to use the discounting rate $r^y + b$, where r^y is the regular USD -discounting rate of the protection buyer and b the aforementioned cross-currency basis spread between both currencies. This is in line with Lemma 2.1, since the spread between the EONIA rate r^C and the protection



buyer's *EUR*-discounting rate r^x is precisely the cross-currency basis spread $b = r^C - r^x$.

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