MECHANICS OF A TRANCHE CDS AFTER A CREDIT EVENT IN THE UNDERLYING BASKET

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Abstract We assume the reader is familiar with the mechanics of tranche CDSs, which we have explained in several earlier articles, e.g. in Mai (2014). In the present article we provide a concise addendum focusing on the explanation of the mechanics when pricing and hedging tranche CDSs after a credit event in the basket. Notice that consideration of one default is sufficient, because the described procedure can simply be iterated when further defaults occur.

1 Some trading conventions

Figure 1 depicts pricing runs for index tranches on the Xover Series 32 index, before and after one default (of the constituent Hema BV) has already been observed in the index. Notice that all potential market factors that have a pricing effect (reference index spread, correlations) have changed only a little bit between these two runs, so the fundamental difference is really only the default event. Furthermore, the outcome of the CDS auction with respect to the default event was anticipated, so there was no essential difference between realized recovery and CDS price prior to default, i.e. no noteworthy PnL “jump” in a held tranche position but only a shift from market price into cash.

Xover S32

	5Y Dec24 (305)	Chg	Delta	Cpn	Indic Spd
0-10	58.70 / 59.70	+.30	3.00	500	2861
10-20	4.80 / 5.60	+.10	2.80	500	636
20-35	-13.15 / -12.45	-.10	1.30	500	196
35-100	58.50 / 65.50	-.5	0.36	500	62

Index Cpn: 500 Index Factor: 1.000 Realised Loss: 0.00% Exp Loss: 12.01%
Equity Tranche Factor: 1.000 SS Tranche Factor: 1.000

Xover S32

	5Y Dec24 (295)	Chg	Delta	Cpn	Indic Spd
0-10	56.65 / 58.25	+.10	2.90	500	2748
10-20	5.05 / 6.15	+.10	2.80	500	647
20-35	-13.45 / -12.65	-.10	1.30	500	190
35-100	57.5 / 63.5	-.5	0.37	500	61

Index Cpn: 500 Index Factor: 0.987 Realised Loss: 0.42% Exp Loss: 11.64%
Equity Tranche Factor: 0.958 SS Tranche Factor: 0.986

Fig. 1: Run for iTraxx XOVER S32 on 8 (top) and 10 (bottom) September 2020, before and after default of HEMA BV with recovery rate 68.5%.

1.1 Use of the given information

The given information is explained as follows:

- The **realized loss** changes from 0% to 0.42%. This number is computed by the formula $(1-R)/d$, where $R = 68.5\%$ equals the HEMA BV recovery rate and $d = 75$ equals the initial number of index constituents (including HEMA BV). This is the percentage of the index nominal that is already defaulted.

The number $1/d = 1/75$ equals precisely the initial index-weight of HEMA BV.

- The **index factor** changes from 1 to 0.987, where 0.987 is simply the rounded value of $1 - 1/d$ with $d = 75$. It equals the reduced index nominal, since one constituent with weight $1/75$ is gone. This factor is required as follows: the price of the index after the default is computed as the product of the upfront (associated with the given reference spread) times the index factor times the original nominal.
- The **equity tranche factor** changes from 1 to 0.958, where 0.958 is computed by the formula $(10\% - 0.42\%)/10\%$, which is the difference of the original tranche thickness and the realized loss divided by the original tranche thickness. This equity factor is required as follows: the price of the equity tranche after the default is given as the quoted equity tranche upfront times the equity tranche factor times the original nominal.

2 How is the pricing affected?

We formally explain how the pricing of a tranche CDS is affected after an observed default.

2.1 Notation

We apply the following notations:

- d current number of constituents.
- ℓ lower tranche attachment point.
- u upper tranche attachment point.
- τ_i random default time of constituent $i \in \{1, \dots, d\}$.
- R_i recovery rate of constituent $i \in \{1, \dots, d\}$.
- w_i weight of constituent $i \in \{1, \dots, d\}$. We denote $\mathbf{w} = (w_1, \dots, w_d)$ and notice that $w_1 + \ldots + w_d = 1$.

2.2 Mechanics

The essential observation is the following. We observe that the cumulative loss per tranche nominal until time t is given by

$$L^{\ell,u}_{d,\mathbf{w}}(t) = \frac{\min\Big\{\max\Big\{0, \sum_{i=1}^{d} w_i\,(1 - R_i)\,1_{\{\tau_i \le t\}} - \ell\Big\}, u - \ell\Big\}}{u - \ell},$$

and the remaining tranche nominal is given by

$$N^{\ell,u}_{d,\mathbf{w}}(t) = (u - \ell)\,\big(1 - L^{\ell,u}_{d,\mathbf{w}}(t)\big).$$

Now we assume (without loss of generality when suitably relabeling) that the first default occurs at time t and equals τ_d, i.e. $\tau_d = \min\{\tau_1, \dots, \tau_d\} = t$. Introducing the notations[1]

$$\tilde{\ell} := \frac{\ell - (1 - R_d)\,w_d}{1 - w_d}, \quad \tilde{u} := \frac{u - (1 - R_d)\,w_d}{1 - w_d},$$
$$\tilde{w}_i := \frac{w_i}{1 - w_d}, \text{ for } i = 1, \dots, d - 1,$$

[1] Notice that $\tilde{w}_1 + \ldots + \tilde{w}_{d-1} = 1$

it is a matter if simple algebra to observe

$$(u-\ell)\, L^{\ell,u}_{d,\mathbf{w}}(t) = (1-w_d)\,(\tilde{u}-\tilde{\ell})\, L^{\tilde{\ell},\tilde{u}}_{d-1,\tilde{\mathbf{w}}}(t), \qquad (1)$$

$$N^{\ell,u}_{d,\mathbf{w}}(t) = (1-w_d)\, N^{\tilde{\ell},\tilde{u}}_{d-1,\tilde{\mathbf{w}}}(t). \qquad (2)$$

Since the random net present value of all cash flows in a tranche CDS is homogeneous of order one[2] in $\big((u-\ell)\, L^{\ell,u}_{d,\mathbf{w}}(t),\, N^{\ell,u}_{d,\mathbf{w}}(t)\big)$, this implies that the remaining tranche CDS after the first default at t is equivalent to a tranche CDS without observed defaults, $d-1$ constituents, new weights $\tilde{\mathbf{w}}$, new attachment points $\tilde{\ell}$ and $\tilde{u}$, and nominal reduced by the factor $1-w_d$.
Notice that for $\ell = 0$ the new lower attachment point $\tilde{\ell}$ is negative, but we see in this case that the inner maximum can be dispensed with and

$$(u-\ell)\, L^{0,u}_{d,\mathbf{w}}(t) = (1-w_d)\,(\tilde{u}-0)\, L^{0,\tilde{u}}_{d-1,\tilde{\mathbf{w}}}(t) + (1-R_d)\, w_d,$$

$$N^{0,u}_{d,\mathbf{w}}(t) = (1-w_d)\, N^{0,\tilde{u}}_{d-1,\tilde{\mathbf{w}}}(t). \qquad (3)$$

If we do not like the interpretation of a tranche with negative lower attachment point, we can thus alternatively define $\tilde{\ell} := 0$ in this case and apply the same interpretation as above, with the only difference being the blue summand. The latter intuitively represents the cash amount that is received and booked once at time t and no longer considered in the future. In other words, every analysis after t regarding pricing and hedging should ignore the blue summand completely and work with the new parameters $\tilde{\ell}$, $\tilde{u}$, $\tilde{\mathbf{w}}$, as well as a reduced nominal by the factor $1-w_d$. Notice that this logic holds for all tranches as well as for the index itself (which might be considered to be a tranche with $\ell = 0$ and $u = 1$).

2.3 Now what happens precisely?

We denote by $f = f\big((u-\ell)\, L, N\big)$ the (random) net present value of all cash flows of the contract with L and N denoting loss and remaining nominal. Recall that f is homogeneous of order one. We denote by p_b the upfront of this contract before the default, and by p_a the upfront after the default (as quoted). By definition, this means[3]

$$p_b = \frac{1}{u-\ell}\, \mathbb{E}\Big[f\Big((u-\ell)\, L^{\ell,u}_{d,\mathbf{w}},\, N^{\ell,u}_{d,\mathbf{w}}\Big)\Big]$$

$$p_a = \frac{1}{\tilde{u}-\tilde{\ell}}\, \mathbb{E}\Big[f\Big((\tilde{u}-\tilde{\ell})\, L^{\tilde{\ell},\tilde{u}}_{d-1,\tilde{\mathbf{w}}},\, N^{\tilde{\ell},\tilde{u}}_{d-1,\tilde{\mathbf{w}}}\Big)\Big].$$

At default time t with the help of (1) we observe for non-equity tranches the equality

$$p_b = (1-w_d)\, \frac{\tilde{u}-\tilde{\ell}}{u-\ell}\, p_a = p_a,$$

since $(1-w_d)\,(\tilde{u}-\tilde{\ell}) = u-\ell$. Thus, the default event has no effect on non-equity tranches. In comparison, for the equity

[2] A real-valued function $f(x_1, x_2)$ in two variables is homogeneous of order one if $f(t\, x_1, t\, x_2) = t\, f(x_1, x_2)$ holds for arbitrary $t > 0$.
[3] We omit the time argument t here in order to simplify notation.

tranche (with $\ell = \tilde{\ell} = 0$), we observe with the help of (3) that

$$p_b = \underbrace{\frac{1}{u}\, w_d\,(1 - R_d)}_{\text{default payment}} + \underbrace{(1 - w_d)\,\frac{\tilde{u}}{u}}_{=\frac{u-(1-R_d)\,w_d}{u}}\; p_a.$$

The reduced equity tranche nominal $(1 - w_d)\,\tilde{u}/u$ is precisely the equity tranche factor, as given in the bottom run of Figure 1. Consequently, the newly quoted price p_a refers to unit notional of the "new" equity tranche with adjusted attachment points and reduced basket, but the nominal of the original contract is reduced by the equity tranche factor. For instance, if we had 20 EUR equity tranche contract, after the default we only have 19.16 EUR nominal outstanding, and

$$20\,\times\,p_b = 0.84 + 19.16\,\times\,p_a,$$

where 0.84 EUR equals the default payment.

3 Summary

Summarizing, when pricing or analyzing tranche or index CDS contracts further after an observed default, the following adjustments have to be done:

(i) Reduce the nominal of your existing index positions by the index factor.

(ii) Reduce the nominal of your existing equity tranche positions by the equity tranche factor.

(iii) Compute new attachment points (in tranches), compute new weights and reduce number of constituents (in index and tranches). The quoted upfronts (or par spreads) are prices for these contracts with unit nominal (original contracts mutate into these).

References

J.-F. Mai, Tranche round trip: dependence matters!, *XAIA homepage article* (2014).