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SIMULATING THE PRICE EVOLUTION OF A CONVERTIBLE BOND IN THE JDCEV MODEL

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Abstract Based on the availability of a tree pricing algorithm for an American-style derivative, the present note demonstrates how the underlying pricing tree can be used efficiently to generate the future price evolution of the derivative contract. The method is explained along the particular case of a convertible bond within the so-called JDCEV model, which is a popular defaultable Markov diffusion model.

1 Introduction

The simulation of the path of a derivative can be desired within a risk management investigation, such as the computation of a Value-at-Risk. The most common pricing approaches for American-style derivative contracts are finite difference pricing and tree pricing methods. Loosely speaking, both methodologies perform equally well with regards to computational speed, and are typically significantly more efficient than more time-consuming Monte Carlo pricing methods. In both methods the numerical evaluation of derivative prices is accomplished by a backward iteration in time. When derivative prices are stored at all time steps, stochastic simulation of a price path can be accomplished from these data by a forward iteration, which is demonstrated in the present note.

For educational purpose, we demonstrate the simulation of an American-style derivative in a particular case, namely that of a convertible bond. To this end, we recall the so-called jump-to-default extended constant elasticity of variance (JDCEV) pricing model of Carr, Linetsky (2006). We demonstrate how to construct a tree pricing algorithm for this model based on the generic recipe of (Brigo, Mercurio , 2006, Appendix F), which is extremely accurate, and then illustrate how this tree approximation can be used for the simulation of convertible bond price paths. For the JDCEV model we prefer the presented tree pricing method over the more typical finite difference procedure, because in our experience it is more accurate in general.

The remaining article is organized as follows. Section 2 demonstrates the construction of a tree pricing algorithm in the JDCEV model. Section 3 shows how the tree pricing model can be used for path simulation of a convertible bond. Section 4 concludes.

2 Tree pricing in the JDCEV model

The idea of tree pricing is to replace a continuous-time model by a discrete-time model with finite state space, for which simpler pricing algorithms are available. The price of the discrete-time model is then used as approximation for the desired price of the continuous-time model. Paragraph 2.1 reviews the trinomial tree construction method of (Brigo, Mercurio , 2006, Appendix F). Paragraph 2.2 demonstrates how to apply this methodology to the JDCEV model. Paragraph 2.3 shows how to use this discrete-time model to compute derivative prices.

2.1 Tree construction

The continuous-time share price process $\{S_t\}_{t\in[0,T]}$ is approximated by a sequence $\{S_i^{(m)}\}_{i=0,1,\ldots,m}$, $m \in \mathbb{N}$, of discrete-time stochastic processes with the following properties:

(a) the random variable $S_i^{(m)}$ takes values in a finite set, for each $i = 0, \ldots, m$, $m \in \mathbb{N}$.

(b) there is a sequence of partitions $0 = t_0^{(m)} < t_1^{(m)} < \ldots < t_m^{(m)} = T$, $m \in \mathbb{N}$, with $\max_{i=1,\ldots,m} |t_i^{(m)} - t_{i-1}^{(m)}|$ converging to zero as $m \to \infty$, such that

$$\left(S_0^{(m)}, \ldots, S_m^{(m)}\right) \overset{d}{\approx} \left(S_0, S_{t_1^{(m)}}, \ldots, S_{t_{m-1}^{(m)}}, S_T\right),$$

with the approximation becoming an equality as $m \to \infty$.

In the classical Black-Scholes stock price model, such discrete-time approximations are well-studied and their validity can be derived rigorously using Donsker's Theorem, which makes precise in which sense a discrete-time random walk converges to Brownian motion, cf. Müller (2009) for a nice summary. For more general diffusions, in particular when drift or volatility become state-dependent, a rigorous justification for the validity of such an approximation is very hard in general. Under boundedness assumptions on the drift and volatility, the so-called Kushner Theorem provides such a justification in some cases. However, in practice these boundedness assumptions are hardly ever met. Luckily, however, there exist many heuristic approaches to derive such discrete-time approximations, whose implementations can be back-checked against closed formulas for European-style derivatives in order to empirically justify their use. This is what most practitioners do.
One particular such approach can be found in (Brigo, Mercurio , 2006, Appendix F). It is designed for diffusions of the form

$$\mathrm{d}D_t = \mu(D_t, t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t \tag{1}$$

without state-dependent volatility, and in the following we briefly sketch the method. The technique is not restricted to diffusions without state-dependent volatility, but becomes feasible if there is a function F such that the driving diffusion of a reduced-form credit-equity model allows to be represented as $Z_t = F(D_t)$, with D_t from (1) having no state-dependent volatility. This is typically the case in concrete models, an example is provided below in paragraph 2.2.
A discrete-time approximation $(D_0^{(m)}, D_1^{(m)}, \ldots, D_m^{(m)})$ to (1) on the time interval $[0, T]$ is constructed on a time grid $0 = t_0 < t_1 < \ldots < t_m = T$, which is allowed to be arbitrarily fine[1]. We denote $\Delta t_i := t_{i+1} - t_i$ for $i = 0, \ldots, m-1$. We set $D_0^{(m)} := D_0$ for the deterministic start value of the (approximating) process. At each time point t_i, $i = 1, \ldots, m$, we consider a finite number of possible states $d_{j,i} := j\,\Delta d_i$, $j = \ldots, -2, -1, 0, 1, 2, \ldots$, for

[1] We omit the superscript (m) here and in the sequel in order to simplify notation, i.e. $t_i = t_i^{(m)}$ actually depends on m, but we do not explicitly display this dependence. The same holds for the tree nodes $d_{j,i}$ as well as for the tree probabilities p_u, p_m, p_d, which all depend on m.

the approximating random variable $D_i^{(m)} \overset{d}{\approx} D_{t_i}$. The state grid at time point t_i is equidistant with time point-dependent grid width Δd_i, yet to be determined. Applying Euler discretization to (1) we find discrete-time approximations for mean and volatility of the diffusion, which we denote by

$$M_{j,i} := d_{j,i} + \mu(d_{j,i}, t_i)\,\Delta t_i \approx \mathbb{E}^{\mathbb{Q}}[D_{t_{i+1}} \mid D_{t_i} = d_{j,i}],$$
$$V_i := \sigma(t_i)\,\sqrt{\Delta t_i} \approx \sqrt{\mathsf{Var}^{\mathbb{Q}}[D_{t_{i+1}} \mid D_{t_i} = d_{j,i}]}.$$

It turns out convenient to define the grid as $\Delta d_{i+1} := V_i\,\sqrt{3}$, $i = 0, \ldots, m-1$. The idea of the approximation is visualized in Figure 1. Assuming at time step i we are in node $D_i^{(m)} = d_{j,i}$, we determine an index $k = k(j,i)$ such that $D_{i+1}^{(m)}$ takes one of the three values $d_{k+1,i+1}$, $d_{k,i+1}$ or $d_{k-1,i+1}$, with respective probabilities $p_u = p_u(j,i)$, $p_m = p_m(j,i)$ and $p_d = p_d(j,i)$. More precisely, $k(j,i)$ is chosen as the closest integer to $M_{j,i}/\Delta d_{i+1}$. Since there are at most three possible nodes that can be reached from a given node $d_{j,i}$ in each time step, such a tree approximation is also called *trinomial tree*.

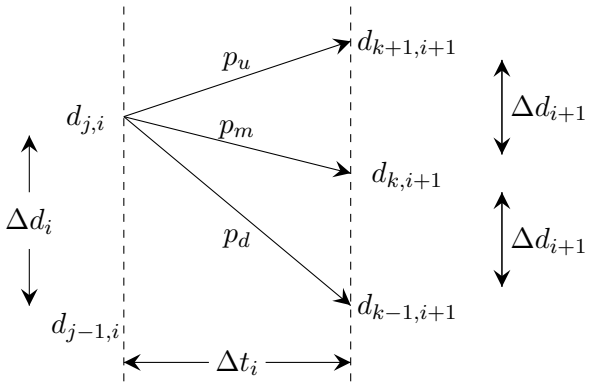

Fig. 1: Visualization of the tree construction.

The three probabilities p_u, p_m and p_d are chosen in such a way that they sum up to one and the conditional first and second moments of the tree approximation match[2] $M_{j,i}$ and $M_{j,i}^2 + V_i^2$, amounting to the linear equation system

$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ M_{j,i} &= d_{k+1,i+1}\,p_u + d_{k,i+1}\,p_m + d_{k-1,i+1}\,p_d, \\ M_{j,i}^2 + V_i^2 &= d_{k+1,i+1}^2\,p_u + d_{k,i+1}^2\,p_m + d_{k-1,i+1}^2\,p_d. \end{aligned}$$

Using the identities $d_{k\pm1,i+1} = d_{k,i+1} \pm \Delta d_{i+1}$ and $\Delta d_{i+1} = V_i\,\sqrt{3}$, the solution to this equation system is given by

$$\begin{aligned} p_d &= \frac{1}{6} + \frac{1}{6}\left(\frac{M_{j,i} - d_{k,i+1}}{V_i}\right)^2 - \frac{M_{j,i} - d_{k,i+1}}{2\sqrt{3}\,V_i}, \\ p_m &= \frac{2}{3} - \frac{1}{3}\left(\frac{M_{j,i} - d_{k,i+1}}{V_i}\right)^2, \\ p_u &= \frac{1}{6} + \frac{1}{6}\left(\frac{M_{j,i} - d_{k,i+1}}{V_i}\right)^2 + \frac{M_{j,i} - d_{k,i+1}}{2\sqrt{3}\,V_i}. \end{aligned} \tag{2}$$

[2]Note that $M_{j,i}^2 + V_i^2 \approx \mathbb{E}^{\mathbb{Q}}[D_{t_{i+1}}^2 \mid D_{t_i} = d_{j,i}]$.

Due to the clever choices of k and Δd_{i+1} it can be shown that these three probabilities are all non-negative, hence actually define a probability distribution. Notice that these clever choices implicitly rely on the assumption that the volatility of the diffusion is state-independent.
The discrete-time (and discrete-state) stochastic process $D_0^{(m)}$, $D_1^{(m)}$, ..., $D_m^{(m)}$ constructed according to this tree by definition satisfies

$$\mathbb{E}^{\mathbb{Q}}\big[(D_{i+1}^{(m)})^{\nu} \,\big|\, D_i^{(m)} = d_{j,i}\big] \approx \mathbb{E}^{\mathbb{Q}}[D_{t_{i+1}}^{\nu} \,|\, D_{t_i} = d_{j,i}],$$

for $\nu = 1, 2$, $i = 0, \ldots, m-1$, and for the respective j that once appear during the construction as some k. The heuristic idea is that for large m, and accordingly a sufficiently fine time grid, this approximative equality of conditional first and second moments carries over to the approximative distributional equality

$$(D_0^{(m)}, D_1^{(m)}, \ldots, D_m^{(m)}) \overset{d}{\approx} (D_0, D_{t_1}, \ldots, D_{t_{m-1}}, D_T),$$

which is ultimately desired. As already mentioned, it is difficult to derive general and at the same time not too restrictive conditions ensuring this convergence in distribution rigorously. However, in practice one can apply this approximation heuristically and check its validity against closed formulas for a particular model. In most cases of practical interest this heuristic approach seems to work reasonably well and passes all such validity checks.
So far, we have established a trinomial tree approximation for diffusions D of the type (1), but in a credit-equity modeling context we ultimately need a tree approximation for a stock price process of the form $S_t = Z_t\, 1_{\{\tau > t\}}$ with some driving diffusion $Z = \{Z_t\}_{t \geq 0}$ and a default time τ, i.e. S jumps to zero at time $t = \tau$. A tree approximation for $S = \{S_t\}_{t \geq 0}$ can be achieved by means of the technique just presented under the assumption that there exists a diffusion $D = \{D_t\}_{t \geq 0}$ and a smooth function F such that the driving diffusion Z of the respective credit-equity model satisfies $Z_t = F(D_t)$. This assumption is satisfied in most practical cases of interest, a concrete example is provided below in paragraph 2.2.

2.2 Application to the JDCEV model

We demonstrate how to apply the heuristic approach of the preceeding section to derive a trinomial tree approximation for the JDCEV model. To this end, we consider a three-parametric specification of the JDCEV model in Carr, Linetsky (2006). In this model, the stock price process is defined by

$$S_t := Z_t\, 1_{\{\tau > t\}}, \quad t \geq 0,$$

where the driving diffusion Z and the bankruptcy time τ are defined by

$$\mathrm{d}Z_t = Z_t \Big(\big(r(t) - \delta + \lambda(Z_t)\big)\, \mathrm{d}t + \sigma(Z_t)\, \mathrm{d}W_t \Big), \quad Z_0 = S_0,$$

$$\tau = \inf \Big\{ t > 0 : \int_0^t \lambda(Z_s)\, \mathrm{d}s > \epsilon \Big\},$$

$$\lambda(z) = \lambda_0 \left(\frac{z}{S_0}\right)^{2\beta}, \quad \sigma(z) = \sigma_0 \left(\frac{z}{S_0}\right)^{\beta},$$

with parameters $\sigma_0, \lambda_0 > 0$, $\beta < 0$, a constant dividend yield $\delta \geq 0$, a deterministic interest rate $r(t)$, and ϵ a unit exponential random variable independent of the Brownian motion $\{W_t\}_{t\geq 0}$. One of the rather striking properties of the JDCEV model is that the probability distribution of S_t is known in closed form, related to the non-central χ^2-distribution. Indeed, we recall from Carr, Linetsky (2006) that

$$\{Z_t\}_{t\geq 0} \stackrel{d}{=} \Big\{F(D_t)\Big\}_{t\geq 0}, \quad F(x) = \Big(|\beta|\, x\Big)^{\frac{1}{|\beta|}},$$
$$D_t = e^{|\beta| \int_0^t r(s)-\delta(s)\,\mathrm{d}s}\, X_{c(t)}^{\left(\frac{\lambda_0}{|\beta|\,\sigma_0^2}-\frac{1}{2\,|\beta|}\right)},$$

where $\{X_t^{(\nu)}\}_{t\geq 0}$ denotes a Bessel process with index ν, and $c(t) = \frac{\sigma_0^2}{S_0^{2\,\beta}} \int_0^t \exp\Big(-2\,|\beta| \int_0^s r(v) - \delta(v)\,\mathrm{d}v\Big)\,\mathrm{d}s$. It can be shown that the process $\{D_t\}_{t\geq 0}$ is a diffusion of the form (1), i.e. with state-independent volatility term. Indeed,

$$\mathrm{d}D_t = \underbrace{\left(|\beta|\,(r(t)-\delta(t))\,D_t + \frac{\frac{\lambda_0}{|\beta|} - \frac{\sigma_0^2}{2\,|\beta|} + \frac{\sigma_0^2}{2}}{S_0^{2\,\beta}\,D_t}\right)}_{=:\mu(D_t,t)}\mathrm{d}t + \underbrace{\frac{\sigma_0}{S_0^{\beta}}}_{=:\sigma(t)}\,\mathrm{d}W_t.$$

It is insightful to remark that for arbitrary $t > 0$ we have $\lambda_0/S_0^{2\,\beta} = \lambda(Z_t)/Z_t^{2\,\beta}$ and $\sigma_0/S_0^{\beta} = \sigma(Z_t)/Z_t^{\beta}$, which can be used to rewrite the drift of the diffusion D as

$$\begin{aligned}\mu(D_t,t) &= D_t\,|\beta|\Big\{r(t)-\delta(t)+\lambda\Big(\big(|\beta|\,D_t\big)^{\frac{1}{|\beta|}}\Big)\\ &\qquad + \frac{|\beta|-1}{2}\,\sigma\Big(\big(|\beta|\,D_t\big)^{\frac{1}{|\beta|}}\Big)\Big\}\\ &= D_t\,|\beta|\Big\{r(t)-\delta(t)+\lambda(Z_t)+\frac{|\beta|-1}{2}\,\sigma(Z_t)\Big\},\end{aligned}$$

which is a representation independent of λ_0 and σ_0 (but instead depending on $\lambda(Z_t)$ and $\sigma(Z_t)$).
The continuous mapping theorem guarantees that the trinomial tree approximation $(D_0^{(m)}, D_1^{(m)}, \ldots, D_m^{(m)})$ for D carries over to a trinomial tree approximation for Z, i.e.

$$\big(F(D_0^{(m)}), F(D_1^{(m)}), \ldots, F(D_m^{(m)})\big) \stackrel{d}{\approx} (Z_0, Z_{t_1}, \ldots, Z_{t_{m-1}}, Z_T).$$

Secondly, a tree approximation for the stock price S must be retrieved from a tree approximation for Z. To this end, recall that the default time τ by definition may be triggered by two qualitatively different events: either the driving diffusion Z diffuses to zero (and then remains there), or the integrated default intensity process $\int_0^t \lambda(Z_s)\,\mathrm{d}s$ reaches the trigger variable ϵ (inducing a share price jump to zero). When deriving a tree approximation for the share price process, a diffusion of Z to zero may simply be taken into account by replacing Z_t with $\max\{Z_t, 0\}$ in the following computations. We observe with an arbitrary bounded and

continuous function $h : [0, \infty) \to \mathbb{R}$ that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[h(S_{t+\Delta}) \,|\, S_t = s,\, \tau > t] \\
&= \mathbb{E}^{\mathbb{Q}}[h(Z_{t+\Delta})\, 1_{\{\tau > t+\Delta\}} \,|\, S_t = s,\, \tau > t] \\
&\qquad + \mathbb{E}^{\mathbb{Q}}[h(0)\, 1_{\{\tau \le t+\Delta\}} \,|\, S_t = s,\, \tau > t] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[h(\max\{Z_{t+\Delta}, 0\})\, e^{-\int_t^{t+\Delta} \lambda(Z_s)\,\mathrm{d}s} \,\Big|\, S_t = s,\, \tau > t\Big] \\
&\qquad + \mathbb{E}^{\mathbb{Q}}\Big[h(0)\, \Big(1 - \, e^{-\int_t^{t+\Delta} \lambda(Z_s)\,\mathrm{d}s}\Big) \,\Big|\, S_t = s,\, \tau > t\Big].
\end{aligned}$$

Taking into account that $\Delta > 0$ in the last computation is a very small time step, we introduce no severe error when assuming that $\lambda(Z_s)$ is identically equal to $\lambda(Z_t)$ for all $s \in [t, t+\Delta]$ (so-called *freezing technique*). This yields the approximation

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[h(S_{t+\Delta}) \,|\, S_t = s,\, \tau > t] \\
&\approx e^{-\lambda(s)\,\Delta}\, \mathbb{E}^{\mathbb{Q}}[h(\max\{Z_{t+\Delta}, 0\}) \,|\, S_t = s,\, \tau > t] \\
&\qquad + \Big(1 - e^{-\lambda(s)\,\Delta}\Big)\mathbb{E}^{\mathbb{Q}}[h(0) \,|\, S_t = s,\, \tau > t].
\end{aligned}$$

Finally, we apply this approximative equality to a time step $t = t_i$ of the tree approximation for D, with $\Delta = \Delta t_i$ and $s = F(d_{j,i})$ for some node $d_{j,i}$, with the notation $p_s(j,i) := \exp(-\lambda(F(d_{j,i}))\,\Delta t_i)$ obtaining

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[h(S_{t_{i+1}}) \,|\, S_{t_i} = F(d_{j,i}),\, \tau > t_i] \qquad (3)\\
&\quad \approx p_s(j,i)\, \mathbb{E}^{\mathbb{Q}}\big[h\big(\max\big\{F(D_{i+1}^{(m)}), 0\big\}\big) \,\big|\, D_i^{(m)} = d_{j,i},\, \tau > t_i\big] \\
&\qquad + \big(1 - p_s(j,i)\big)\mathbb{E}^{\mathbb{Q}}[h(0) \,|\, D_i^{(m)} = d_{j,i},\, \tau > t_i] \\
&\quad = p_s(j,i)\, \Big\{p_u(j,i)\, h\big(\max\big\{F(d_{k(j,i)+1,i+1}), 0\big\}\big) \\
&\qquad + p_m(j,i)\, h\big(\max\big\{F(d_{k(j,i),i+1}), 0\big\}\big) \\
&\qquad + p_d(j,i)\, h\big(\max\big\{F(d_{k(j,i)-1,i+1}), 0\big\}\big)\Big\} \\
&\qquad + (1 - p_s(j,i))\, \mathbb{E}^{\mathbb{Q}}[h(0) \,|\, D_i^{(m)} = d_{j,i},\, \tau > t_i].
\end{aligned}$$

Inductively, assume that we have already defined the trinomial tree approximation $(S_0^{(m)}, \ldots, S_i^{(m)})$ until time t_i, with the induction start $S_0^{(m)} := S_0$ being trivial. We can now construct $S_{i+1}^{(m)}$ by defining its probability distribution conditional on the event $S_i^{(m)} = s$ for the finitely many potential values s that $S_i^{(m)}$ can take with positive probability. By construction, all these values s will be of the form $s = s_{j,i} = F(d_{j,i})$ for some node $d_{j,i}$ of the tree approximation for the diffusion D. Trivially, conditioned on $S_i^{(m)} = 0$, we have that $S_{i+1}^{(m)} = 0$ as well. Conditioned on $S_i^{(m)} = s_{j,i} > 0$, the computation (3) shows that it is reasonable to define the approximating random variable $S_{i+1}^{(m)}$ according to the probability law

$$\begin{aligned}
\mathbb{Q}\big(S_{i+1}^{(m)} = \max\big\{F(d_{k(j,i)+1,i+1}), 0\big\} \,|\, S_i^{(m)} = s_{j,i}\big) &=: p_s(j,i)\, p_u(j,i), \\
\mathbb{Q}\big(S_{i+1}^{(m)} = \max\big\{F(d_{k(j,i),i+1}), 0\big\} \,|\, S_i^{(m)} = s_{j,i}\big) &=: p_s(j,i)\, p_m(j,i), \\
\mathbb{Q}\big(S_{i+1}^{(m)} = \max\big\{F(d_{k(j,i)-1,i+1}), 0\big\} \,|\, S_i^{(m)} = s_{j,i}\big) &=: p_s(j,i)\, p_d(j,i), \\
\mathbb{Q}\big(S_{i+1}^{(m)} = 0 \,|\, S_i^{(m)} = s_{j,i}\big) &=: 1 - p_s(j,i). \qquad (4)
\end{aligned}$$

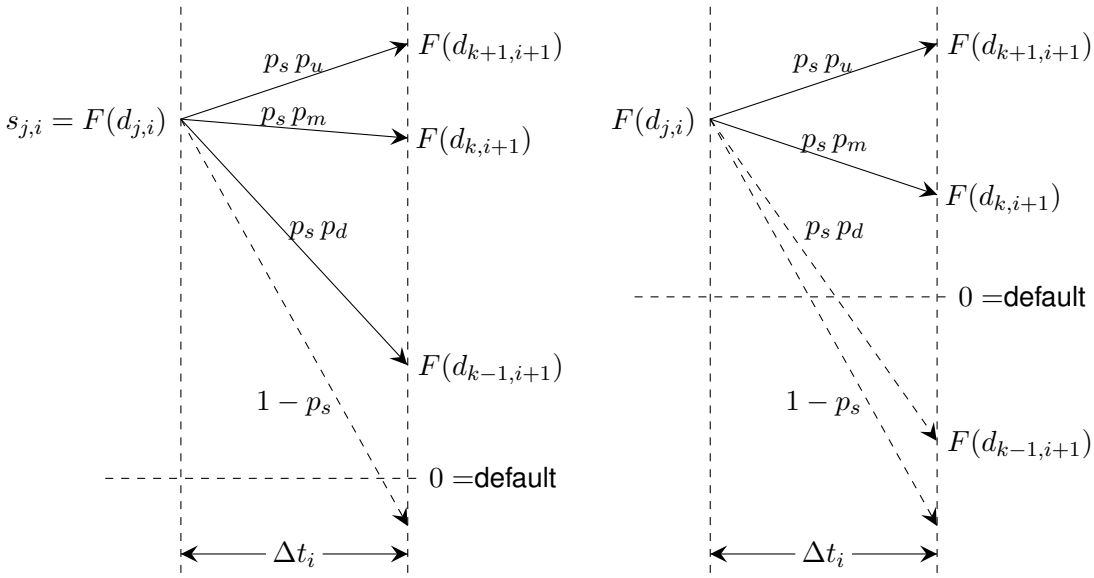

Fig. 2: Visualization of the tree construction for the share price. Left: all three nodes $d_{\ell,i+1}$, $\ell \in \{k+1, k, k-1\}$, that the diffusion D can reach from $d_{j,i}$ in its trinomial tree approximation, satisfy $F(d_{\ell,i+1}) > 0$, and hence become surviving nodes for the share price. Right: one of the three nodes $d_{\ell,i+1}$, $\ell \in \{k+1, k, k-1\}$, becomes a default node in the share price tree (i.e. Z diffuses to zero between t_i and t_{i+1}).

Figure 2 visualizes this construction principle.
Figure 3 visualizes the increasing quality of the tree approximation as the mesh of the time grid tends to zero in the present example. The discrete distributions in the plots arise from a tree approximation on the interval $[t, t+\Delta]$ which is subdivided into 1, 10, 25, respectively 50 equidistant time steps. It is observed how the approximating, discrete probability distribution contains more and more atoms with increasing number of discretization steps.

Remark 2.1 (Flexible time grid)
There also exist different tree approximation methods than the one presented here. However, some of them are based on an equidistant time grid. One of the striking features of the presented tree construction is the fact that the time grid $0 = t_0 < t_1 < \ldots < t_m = T$ needs not be equidistant, i.e. $\Delta t_i \neq \Delta t_j$ is possible for some $i \neq j$. This can be quite convenient in pricing applications, because it allows to insert time points of particular importance into the time grid without further effort. A typical implementation is to start with an equidistant time grid in a first step, and in a second step one adds special time points to this equidistant grid. Such special time points can be dividend payment dates, coupon payment dates, or time points at which certain conversion, call or put periods of a bond start or end.

2.3 Tree pricing via backwardation

In the two preceding paragraphs we have managed to construct a discrete-time and discrete-state approximation $\big(S_0^{(m)}, \ldots, S_m^{(m)}\big)$ for $(S_0, S_{t_1^{(m)}}, \ldots, S_{t_{m-1}^{(m)}}, S_T)$. Now we wish to use this approximating model in order to (approximately) compute the value of

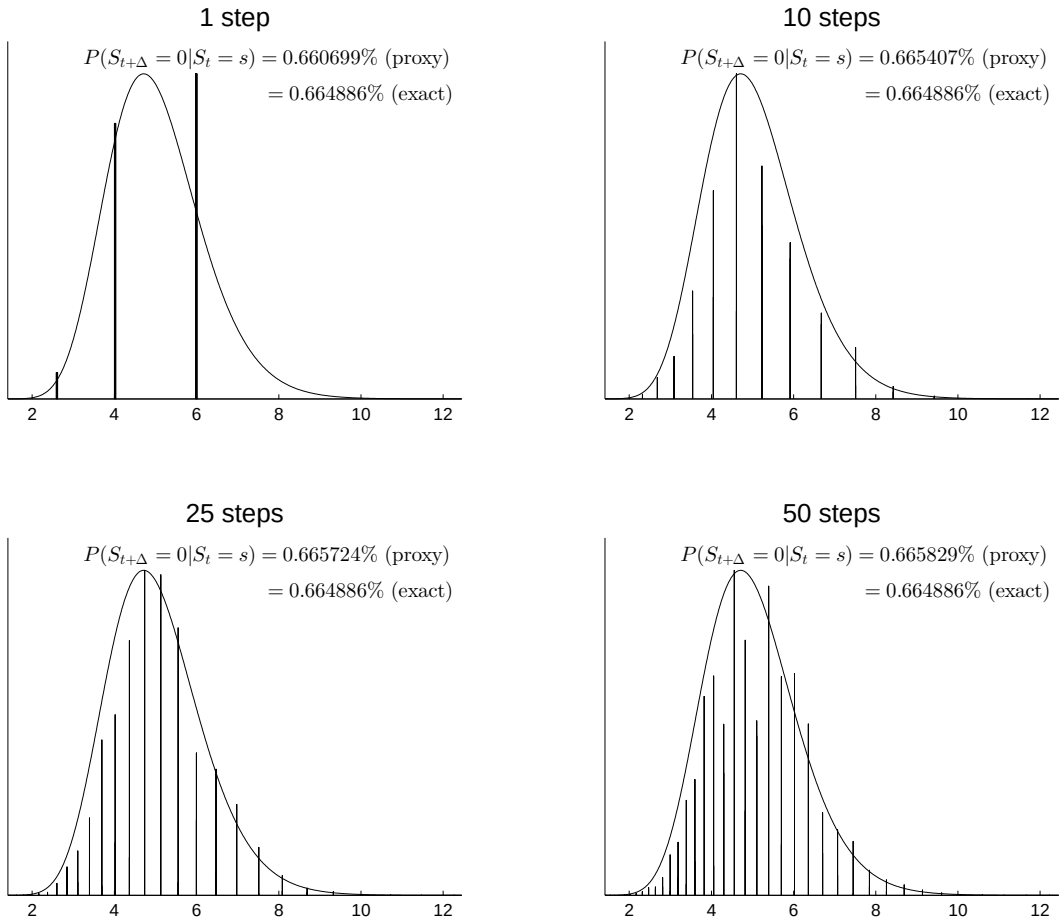

Fig. 3: Visualization of the increasing quality of the tree approximation as the mesh of the time grid tends to zero in the JDCEV model. The chosen parameters are $\lambda_0 = 0.02$, $\sigma_0 = 0.4$, $\beta = -0.2$, $r - \delta \equiv -0.01$, $S_0 = 100$. The depicted density on $(0, \infty)$ equals that of $S_{t+\Delta}$ conditioned on the event $\{S_t = s\}$ with $\Delta = 0.1$, $s = 5$, and $t = 2$, which can be computed in closed form using the connection to the non-central χ^2-distribution derived in Carr, Linetsky (2006). The default probability, which is not contained in the density and omitted as an extra bar at the value 0 in the approximating distribution, is depicted as text in the figures for the sake of visibility.

a derivative security referring to the share price. To begin with, we define a discrete approximation of the default time τ from our approximating model, namely

$$\tau^{(m)} := \min\{t_i^{(m)} \,:\, i = 0, \ldots, m,\, S_i^{(m)} = 0\} \cup \{\infty\}.$$

Considering a European claim $h(S_T)$ on the stock price, the essential idea is to approximate the associated derivative price at

the time points $t_i^{(m)}$, $i = 0, \dots, m$, as

$$\begin{aligned} V_{t_i^{(m)}} &= DF(t_i^{(m)}, T)\, \mathbb{E}^{\mathbb{Q}}\big[h(S_T) \,\big|\, \mathcal{F}_{t_i^{(m)}}\big] \\ &\approx DF(t_i^{(m)}, T)\, \mathbb{E}^{\mathbb{Q}}\big[h(S_m^{(m)}) \,\big|\, \mathcal{F}_{t_i^{(m)}}\big] =: V_i^{(m)}, \end{aligned}$$

where $DF(a,b) := \exp(-\int_a^b r(s)\,\mathrm{d}s)$ denotes a discount for $b \geq a \geq 0$. We denote the finite set of all possible values of $S_i^{(m)}$ by $\mathcal{S}_i^{(m)}$, and for $i = 0, \dots, m-1$ we furthermore introduce the notation

$$q_{i,s_i,s_{i+1}}^{(m)} := \mathbb{Q}(S_{i+1}^{(m)} = s_{i+1} \,|\, S_i^{(m)} = s_i),\ s_i \in \mathcal{S}_i^{(m)},\ s_{i+1} \in \mathcal{S}_{i+1}^{(m)}.$$

These transition probabilities $q_{i,s_i,s_{i+1}}^{(m)}$ are known from the tree construction, see (4). Markovianity of the share price process implies that the random variable $V_i^{(m)}\, 1_{\{\tau^{(m)} \geq i\}}$ is a function of $(i, S_i^{(m)})$, in particular discrete with at most finitely many values, which we denote by $v_{i,S_i^{(m)}}^{(m)}$. In particular, in case of a convertible bond with recovery rate $R \in [0,1]$ the values

$$\begin{aligned} v_{m,S_m^{(m)}}^{(m)} &= h(S_m^{(m)})\, 1_{\{\tau^{(m)} \geq m\}} \\ &= H(S_m^{(m)})\, 1_{\{S_m^{(m)} > 0\}} + R\, 1_{\{S_m^{(m)} = 0\}} \end{aligned}$$

can be directly computed from the discrete probability distribution of $S_m^{(m)}$, namely

$$v_{m,s_m}^{(m)} = H(s_m)\, 1_{\{s_m > 0\}} + R\, 1_{\{s_m = 0\}}, \quad s_m \in \mathcal{S}_m^{(m)}. \tag{5}$$

Here, $H(x)$ denotes the value of the convertible bond at maturity if the stock price equals x, which is typically the maximum of parity and the final redemption/coupon payment. Our goal is to compute the values $v_{i,s_i}^{(m)}$ for $i < m$ and $s_i \in \mathcal{S}_i^{(m)}$, in particular $v_{0,S_0}^{(m)}$. For $i = 0, \dots, m-1$, the tower property of conditional expectation allows us to compute the set $\{v_{i,s_i}^{(m)}\}_{s_i \in \mathcal{S}_i^{(m)}}$ from knowledge about the set of values $\{v_{i,s_{i+1}}^{(m)}\}_{s_{i+1} \in \mathcal{S}_{i+1}^{(m)}}$, as

$$\begin{aligned} v_{i,s_i}^{(m)} &= 1_{\{s_i > 0\}}\, DF(t_i^{(m)}, T)\, \mathbb{E}^{\mathbb{Q}}\big[h(S_m^{(m)}) \,\big|\, S_i^{(m)} = s_i\big] \\ &= 1_{\{s_i > 0\}}\, DF(t_i^{(m)}, t_{i+1}^{(m)}) \times \\ &\quad \times\, \mathbb{E}^{\mathbb{Q}}\big[DF(t_{i+1}^{(m)}, T)\, \mathbb{E}^{\mathbb{Q}}\big[h(S_m^{(m)}) \,\big|\, \mathcal{F}_{t_{i+1}^{(m)}}\big] \,\big|\, S_i^{(m)} = s_i\big] \\ &= 1_{\{s_i > 0\}}\, DF(t_i^{(m)}, t_{i+1}^{(m)})\, \mathbb{E}^{\mathbb{Q}}[V_{i+1}^{(m)} \,|\, S_i^{(m)} = s_i] \\ &= 1_{\{s_i > 0\}}\, DF(t_i^{(m)}, t_{i+1}^{(m)}) \sum_{s_{i+1} \in \mathcal{S}_{i+1}^{(m)}} q_{i,s_i,s_{i+1}}^{(m)}\, v_{i+1,s_{i+1}}^{(m)}. \end{aligned} \tag{6}$$

Together with (5), Equation (6) yields a recursion for the computation of $v_{i,s_i}^{(m)}$ for all $i = 0, \dots, m$ and $s_i \in \mathcal{S}_i^{(m)}$. Since this recursion works backward in time, starting from $i = m$, and then iterating over $i = m-1, m-2, \dots$ all the way back to $i = 0$, this pricing methodology is called *backwardation*. Referring to the particular trinomial tree construction of the preceding paragraph, the involved transition probabilities $q_{i,s_i,s_{i+1}}^{(m)}$ are non-zero only for

four values of s_{i+1}, at least one of these four values being equal to zero and representing default.
For American-style derivatives, such as a convertible bond with American conversion option (and possibly also American soft call option), each backward iteration step has to be followed by an updating step that decides between continuation or exercise. More precisely, after $v_{i,s_i}^{(m)}$ has been computed, this value has to be replaced with parity $\alpha\, s_i$ (with α denoting the conversion ratio) in case parity is larger than $v_{i,s_i}^{(m)}$. Similarly, it has to be replaced with a call strike C_i in case it is larger than that, and the call right is valid (which in case of a soft call is the case if s_i is larger than some trigger level). In short, $v_{i,s_i}^{(m)} \rightsquigarrow \max\{\min\{v_{i,s_i}^{(m)}, C_i\}, \alpha\, s_i\}$.

3 Simulation along the pricing tree

The random vector $(V_0^{(m)}, \ldots, V_m^{(m)})$ from the previous paragraph is a well-justified simulation of the price evolution of the convertible bond in concern. It can be simulated based on its possible values $v_{i,s_i}^{(m)}$ and the transition probabilities $q_{i,s_i,s_{i+1}}^{(m)}$. More precisely,the simulation works as follows:

Algorithm 1 (Simulation of the convertible bond price path)
We simulate one realization $(v_0, v_1, \ldots, v_m)$ of the random vector $(V_0^{(m)}, \ldots, V_m^{(m)})$ as follows.

(0) The value $v_0 := V_0^{(m)}$ is non-random and can simply be read off from the tree. Further, set $i := 0$ and $s_0 := S_0$.

(1) While $i < m$, repeat the following steps:

(1.1) Simulate a realization x of the random variable $S_{i+1}^{(m)}$ using the discrete probability law

$$\mathbb{Q}(S_{i+1}^{(m)} = x) = q_{i,s_i,x}^{(m)}, \quad x \in \mathcal{S}_{i+1}^{(m)}.$$

(1.2) Set $v_{i+1} := v_{i+1,x}^{(m)}$, and set $i := i+1$ as well as $s_i := x$.

Figure 4 visualizes simulated paths of a convertible bond that have been generated according to this algorithm. The initial bond price is 90%, the initial stock price equals 7.08 and the conversion ratio is 0.076793119, so that parity equals 54.37%. The bond has a soft call strike at (parity equal to) 130% and the recovery rate is specified as 40%, bond maturity is 20 September 2023. The parameters are chosen such that the bond price is highly volatile, which can be seen in the plot, to wit $r - \delta \equiv 0.01$, $\sigma = 0.875$, $\beta = -0.3$, $\lambda = 0.054$.

4 Conclusion

We have demonstrated the possibility to simulate paths of an American-style derivative contract based on a tree pricing framework. The procedure has been illustrated in the particular case of a convertible bond within the JDCEV model of Carr, Linetsky (2006).

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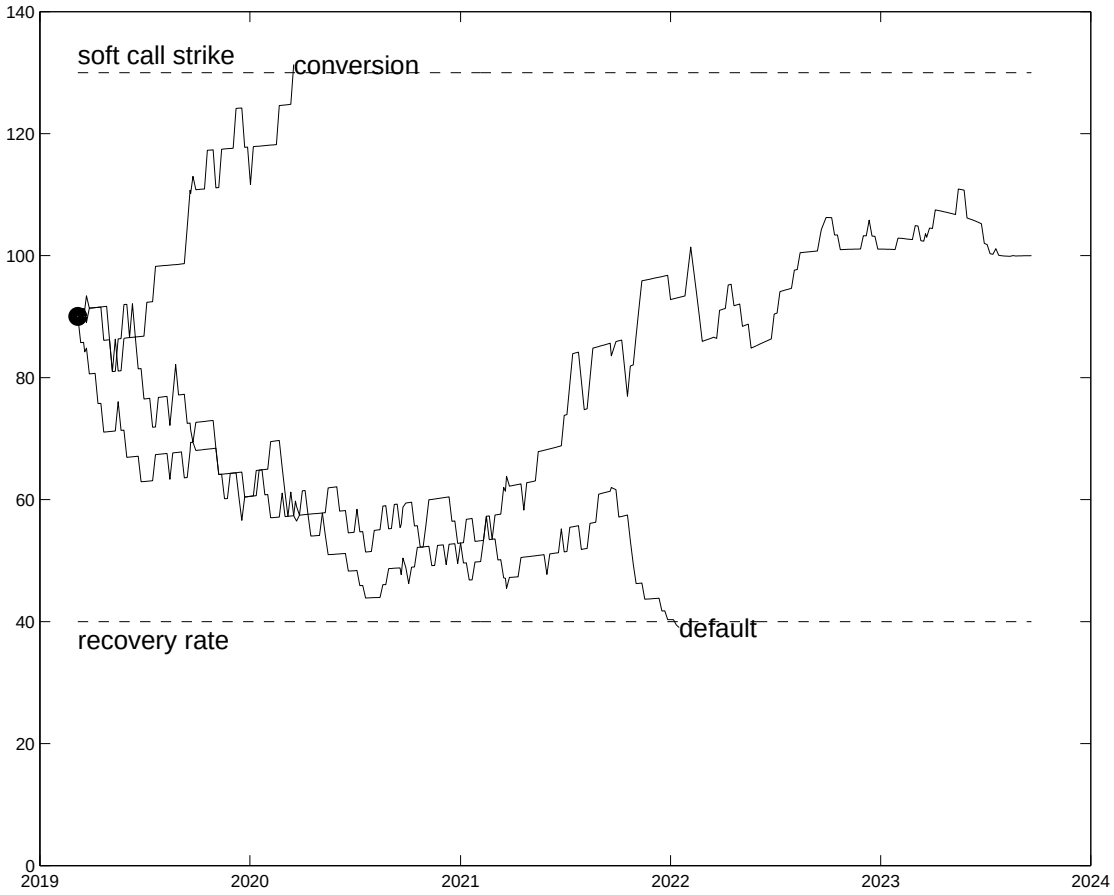

Fig. 4: Visualization of three simulated paths of a certain convertible bond price according to the described methodology.

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