XAIA INVESTMENT

Pricing CoCos with equity conversion feature in a distressed market environment

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Abstract Many contingent convertible bonds (CoCos) issued since 2014 belong to the additional Tier 1 (AT1) capital of the issuing bank and are thus of a perpetual nature. Within a Black-Scholes setup, under the very conservative assumption that the CoCo trigger is activated by an adverse entity, we are able to derive a closed-form expression for the fair price of such an instrument, provided it has an equity conversion feature (no write-down feature). This formula allows for a quick understanding of the mechanics of such AT1 CoCos, helps to efficiently compute delta-hedge ratios, and it can be implemented easily on a spreadsheet. Furthermore, the closed-form solution may be used as an integral building component of an efficient, semi-analytical pricing formula for AT1 CoCos, when an additional call right for the issuing bank and a step up coupon is present, as typical in the marketplace. Finally, we demonstrate that over the last few years observed CoCo market prices were too high to be explained by our adverse entity assumption, but currently (after the recent Credit Suisse CoCo wipe-out in March 2023) have come down to levels that can be modeled reasonably with our approach. Our approach can therefore be a useful tool to deal with CoCo prices in a market environment with distressed banking sector.

1 Introduction

Contingent convertible (CoCo) bonds are issued by European banks in accordance with the Basel III framework[1]. They are debt instruments, often of a hybrid nature, which are intended to serve as a buffer for more senior debt holders in case the bank runs into financial distress. For background knowledge on the structure of liabilities in a bank's balance sheet the reader is referred to Appendix A. Initially, CoCos are similar to subordinate bonds issued by the bank, but they are either written down (partially or fully, temporarily or permanently) or converted into equity upon the occurence of a *trigger event*. A full nominal write down feature is also sometimes called a wipe-out feature. The trigger event can be of various kinds in principal. Currently outstanding CoCos feature a combination of two different types of trigger events:

- *Accounting trigger:*
  - An accounting ratio, typically the CET1 ratio, falls below a certain threshold, see Appendix A for a definition of CET1.

[1] More specifically, according to the Capital Requirement Regulation (CRR) and Capital Requirement Directive IV (CRD-IV).

- *Point of non-viability (PoNV) trigger:*
  - A specific regulator makes the decision that the trigger event has occurred.

In the current market environment, it is highly likely that an AT1 CoCo breaches the PoNV trigger long before the accounting trigger is hit, because the accounting triggers are set at very low levels which nowadays are viewed as being not only critical but already fatal to the bank. According to Basel III, Pillar 1 defines strict European-wide laws for all banks, whereas Pillar 2 leaves some freedom for national regulators to further refine the latter. For example, the British and Danish central banks have set the CET1 ratio trigger level for the AT1 CoCos issued by their banks at 7%, while the Spanish and Italian central banks have set it at 5.125%. It is furthermore worth noticing that AT1 CoCos typically feature the following covenants, which are in line with the idea of AT1 capital as being “going concern” capital, as opposed to T2 capital as being “gone concern” capital:

- *Discretionary coupon:* The issuing bank may omit coupon payments. More senior bonds (T2 and senior) typically feature a dividend-stop covenant, i.e. when a coupon is not paid, the bank must not pay a dividend to its shareholders. This is not the case for AT1 CoCos, i.e. in theory the bank might pay a dividend on its stock but omit payment of an AT1 CoCo coupon. Even though omitting a coupon on an AT1 CoCo must be “reconciled” with the regulator, and even though it is of course not a popular decision by the bank’s management from the CoCo investors’ point of view, it is still possible in theory, and constitutes a certain risk for the investor.

- *Regulatory call:* The issuing bank may redeem the bonds at par, provided there is a change in the legislation affecting future CoCo covenants. For instance, if the regulator decides that the CET1 ratio trigger level for newly issued CoCos changes from 7% to 8%, the issuing bank may redeem its outstanding CoCos with the old trigger level at par. Since the half-life of regulatory rules is not too long these days and since many outstanding CoCos trade above par due to the high demand for high-yield bonds in a low interest rate environment, such a regulatory call constitutes a certain risk for the investor.

- *Call and coupon step-up:* After a period in which the CoCo pays a fix coupon rate c_1, the coupon rate changes to a floating coupon plus a fix margin. The floating coupon is linked to some benchmark rate, e.g. a 5-year mid-swap rate. The margin to be paid on top of the floating rate is determined as c_1 minus the value of the benchmark rate on the issuing date of the bond. This definition reflects the idea that, provided the benchmark rate remains unaltered, the coupon rate received after the fix coupon period remains the same. However, since the expected forward mid-swap rates in today’s $\$$- or €-swap curves are higher than their today’s values, the change from fix to floating coupon actually constitutes a coupon step-up for the investor. Unfortunately, this does not constitute a benefit for the investor, because after the fix coupon period the issuing bank obtains continuous (or Bermudan) call rights at par, i.e. may at their own will (without consulting the regulator) redeem the CoCos at par. Due to the aforementioned effective coupon step-up, one might argue that there is indeed a strong incentive for the issuer to call the CoCos at par. This feature is therefore slightly irritating as it contradicts the idea of AT1 capital as being "going concern" capital. In the old framework of Basel II, such step-up coupons along with call rights at par were rather characteristic for UT2 bonds.

As a consequence of these characteristic features of AT1 covenants, we conclude that an AT1 CoCo should rather be viewed as a "trust investment" into the issuing bank than as a debt instrument. One might therefore argue that it somehow is closer to equity than to debt. Formally, before the trigger event a CoCo might be classified as a T1-bond or a T2-bond. According to Glionna et al. (2014), in terms of the bank's capital structure CoCos should actually be thought of as lying between CET1 and AT1, see Appendix A for background on these notions. In the present article, we totally ignore the discretionary coupon feature and the regulatory call feature, simply because these are too "qualitative" to be modeled by quantitative means, and also because the risk for the investors is secondary compared with other involved risks. However, we do tackle the call and coupon step-up covenant, because it is too important to be ignored in our view. Moreover, we consider only perpetual CoCos with full equity conversion feature, because the write-down feature is really a whole different story. To be precise, we consider CoCos for which, upon occurence of the trigger event, the investor receives α shares of the issuing bank, where α is called the *conversion ratio*.

Remark 1.1 (On the structure of the conversion ratio)
For the major part of the bonds, the conversion ratio α is contractually specified. The CoCo prospectus specifies a so-called conversion price, whose reciprocal equals the conversion ratio α. The conversion price is defined as the highest of (i) the stock price at conversion, (ii) some fixed floor price[2], and (iii) the nominal value of one share, which is close to zero and thus neglected in our study. If the share price trades above the floor price at conversion, then the CoCo investor receives par on his notes. However, if the share price trades below the floor price, the conversion ratio equals the reciprocal of the floor price. As conversion is expected to happen at an unfavorable time point from the view of the CoCo holder, the share price at that time presumably lies below the floor price. Therefore we choose the reciprocal of the floor price in the prospectus as a proxy for the conversion ratio α.

1.1 Review of pricing approaches

Generally speaking, the pricing of CoCos is very difficult, not only but mainly due to the fact that the involved trigger event is difficult to incorporate into well-known credit- or equity-derivatives pricing approaches. This is because the fundamental quantities involved (e.g. CET1 ratio or regulator's decision) are difficult to explain jointly with quantities that are usually modeled in standard

[2] Subject to certain amendments in case of capital restructuring events.

approaches, such as stock price or default intensity. A second problem with the pricing of CoCos is the fact that their covenants before the trigger event are often non-standard, as mentioned earlier. Clearly, such complicated covenants complicate the pricing significantly already in the regular non-CoCo bond case. Existing pricing approaches can be classified as follows:

- **Mertonian balance sheet approach:** The idea is to explain the CoCo-issuing bank as a whole, and then derive prices for the CoCo-parts of the respective bank's balance sheet. From a theoretical viewpoint, the structural balance sheet approach seems to be the right way to go. It provides the only suitable setup for a definition of the difficult-to-model trigger event as accurate as possible by means of a mathematical model. From a practical viewpoint, this comes at the cost of having to model many parts of the bank's balance sheet, and hence working with a model whose mechanics are hard to overlook, and which is prone to parameter overflow. To provide examples of major contributions in this direction, Madan, Schoutens (2011a) opt for an ansatz incorporating conic finance[3], an approach nicely described in Madan, Schoutens (2011b). Another structural approach, which basically is an adaptation of the classical debt-equity model of Leland (1994) to CoCos, can be found in Albul et al. (2010). Penacchi (2010) applies a structurally similar approach, but even uses more advanced stochastic processes (stochastic interest rates and jump-diffusion) and solves the model via Monte Carlo algorithms. Similar structural approaches in this direction are formulated in Brigo et al. (2013); Cheridito, Xu (2013). The latter reference additionally discusses a reduced-form model in which both the CoCo trigger event and the issuing bank's bankruptcy time are modeled as the first two jumps of a time-changed Poisson process. Finally, the article Glasserman, Nouri (2012) provides a very nice structural equilibrium modeling approach to the pricing of CoCos.

Unlike the aforementioned references, the present article's goal is to find a shortcut to the CoCo pricing problem. It is clear that this must come at the cost of stringent assumptions, but we still give it a shot and consider a very "reduced" setup with a reasonable trade-off between realism and practical viability. In the existing literature, there is one popular reference with a similar intention, namely De Spiegeleer, Schoutens (2012). Assuming the CoCo bond to be *bullet*, i.e. a regular coupon bond with finite maturity, the authors present two simple-to-implement approaches, whose ideas are briefly summarized:

- **Credit derivatives approach:** The CoCo is priced like a bullet bond, only the usually underlying "credit event" is replaced by the "trigger event", which is assumed to happen before a regular credit event. With this logic, the usual credit spread parameter λ used for the computation of default probabilities (in the simplest credit risk model with constant intensity) is

[3] A theory replacing the usual "law of one price" by bid-ask spreads in order to account for illiquidity effects in the markets.

replaced by a higher trigger spread parameter $\lambda_{Trigger}$. The latter is assumed to be a strictly decreasing function of the underlying stock price, derived from the first hitting probability of a geometric Brownian motion under a certain threshold. In particular, the CoCo trigger event is assumed to be given by the share price triggering this threshold, for which an assumption needs to be made.

- **Equity derivatives approach:** The CoCo is decomposed into a regular bond and a portfolio of certain digital stock options. To this end, it is assumed that in case of a trigger event before the maturity of the CoCo the shares are not received immediately but only at maturity date of the CoCo. This implies that we may write the discounted cashflows of the CoCo with unit nominal, coupon payments c_i at times t_i, $i = 1, \ldots, N$, and final maturity t_N as

$$DF(t_N)\left(1_{\{\tau > t_N\}} + \left((1-\beta) + \beta\,\alpha\,S_{t_N}\right)1_{\{\tau \leq t_N\}}\right) + \sum_{i=1}^{N} DF(t_i)\left(c_i\,1_{\{\tau > t_i\}} + (1-\beta)\,c_i\,1_{\{\tau \leq t_i\}}\right), \qquad (1)$$

where $\beta \in [0,1]$ denotes the contractually specified fraction of nominal to be converted into shares upon the trigger event, τ denotes the arrival time of the trigger event, α the number of shares to be received in case of conversion per unit of CoCo nominal (*conversion ratio*), S_t the share price at time $t \geq 0$, and $DF(t)$ denotes a discount factor for time point $t \geq 0$. Now further assuming that the stock price is given as in the classical Black-Scholes model and defining

$$\tau := \tau_L := \inf\{t > 0 \,:\, S_t \leq L\} \qquad (2)$$

with a model threshold L implies that the expectation value over the expression in (1) can be computed in closed form, see Rubinstein, Reiner (1991) for the respective formulas.

Both approaches of De Spiegeleer, Schoutens (2012) rely on the assumption of the trigger event being defined via (2) as the first hitting of the share price below a threshold, which does not reflect market reality. In particular, the right choice of an appropriate modeling constant L is a task far from trivial. Furthermore, the assumption of a functional relationship between the CET1 ratio and the stock price is highly questionable. Moreover, it has been outlined earlier that it is highly likely that the PoNV trigger is hit before the accounting trigger anyway, which renders the modeling of the CET1 ratio obsolete. Sundaresan, Wang (2010) point out another economic problem with a trigger definition like in (2): at conversion there will usually take place a value transfer between contingent capital investors and equity holders, the direction of the transfer being determined by the conversion ratio. As a consequence of the existence of such a value transfer CoCo and equity are not robust to price manipulation. Unfortunately, Sundaresan, Wang (2010) point out that it is typically impossible to set the conversion ratio a priori at a level avoiding the existence of a value transfer upon conversion.

As a consequence of these considerations, the core idea of the present article is to leave through the back door and completely avoid an explicit definition of the trigger event arrival time τ. In contrast, we make the assumption that τ is chosen by an external entity in such a way that the CoCo investor's expected earning is minimized (*adverse entity assumption*). Appearing peculiar on the first glimpse, we argue that this assumption is not too unrealistic because it reflects the fact that CoCo investors are by definition the weakest link of the chain of debt holders, and the trigger timing is most likely not to be in their favor. Surprisingly, under this assumption τ will be shown to be of the form (2) for a "worst-case" trigger level depending on the CoCo covenants, the interest rate, and the stock volatility.
The remaining article is organized as follows. Section 2 derives a closed formula for perpetual CoCos with equity conversion feature, when no additional call rights for the issuing bank are present. Section 3 extends this result to the case when the CoCo provides its issuing bank with an additional call right (along with a step up coupon), as is the case in practice. Section 4 discusses the derived pricing approach in a concrete real-world example. Finally, Section 5 concludes and the Appendix provides the proofs as well as background on a bank's capital structure.

2 Perpetual CoCos

The present section's goal is to derive a closed-form pricing formula for such perpetual CoCos with equity conversion feature. We must rely on some simplifying assumptions in order to do so, but we try to comment on each of them and give our opinion on how severe it is and whether it may be generalized in potential future research. In our view, the value of a closed-form expression for such complicated instruments is at least twofold: (1) It provides a quick understanding of the instrument mechanics that allows to draw conclusions regarding risk profile and hedging. In particular, it serves as a basis for generalizations, and we provide one example for such an extension in Section 3. (2) It may be implemented by traders in a spreadsheet, which is often a necessary requirement in order to spark investors' interest into a new product. In the sequel, we provide a list of the assumptions we impose.

(A1) **Flat interest rate term structure:** We assume that the evolution of a risk-free bank account used for discounting is given by $t \mapsto e^{r\,t}$ for a short rate parameter $r \geq 0$.

(A2) **Continuous coupon payments:** It is assumed that the coupon rate c is paid continuously, instead of discretely (as is the case in practice).

(A3) **Black-Scholes dynamics:** The stock price process $\{S_t\}_{t\geq 0}$ is assumed to follow the dynamics

$$\mathrm{d}S_t = S_t\,(r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t)$$

under a risk-neutral pricing measure, under which $\{W_t\}_{t\geq 0}$ denotes standard Brownian motion. The natural filtration of $\{W_t\}_{t\geq 0}$ is denoted by $(\mathcal{F}_t)_{t\geq 0}$.

Among these, assumption (A1) is probably the most severe for practical purposes. Clearly, a relaxation to the case of a deterministic, but non-constant short rate would be highly desirable. Unfortunately, however, our derivation hinges on the assumption of a flat interest rate curve. Nevertheless, we provide some advice on how to choose r and how to draw conclusions from our result to the more general case of non-constant interest rates in Remark 2.3(c). Assumption (A3) is required in order to end up with closed formulas, but it is also convenient because the popularity of the Black-Scholes model implies that the resulting formula has a high degree of communicability – in particular to traders without deep background in probability theory. The continuous rather than discrete coupon payments in assumption (A2) basically simplify notation but constitute an acceptable impreciseness in our opinion. However, step-up coupons are ruled out by assumption (A2), see Section 3 for a description of how their incorporation affect the problem (and the solution).

Remark 2.1 (On the level of abstraction)
So far, we have not made any assumption on the credit-worthiness of the CoCo-issuing bank, except implicitly via the constant stock price volatility σ. In fact, we will not do so at all but instead remain within this tiny setup, which is very "reduced" compared with, e.g., balance sheet approaches. Implicitely, we assume that a default of the CoCo is impossible, e.g. it is assumed that the trigger event happens almost surely before a coupon is not paid. In our view, besides the ability to obtain a closed pricing formula, this high level of abstraction has the appealing nature that it focuses on the dependence between the CoCo and the stock price. Keeping in mind the adjacency of both instruments in the bank's capital structure and the fact that many CoCo investors are interested in hedging sensitivities of the CoCo with respect to the underlying stock, the model is narrowed down to the essential quantities.

The most critical aspect in the pricing of CoCos is the definition of the trigger event. Denoting by $\tau \in (0, \infty]$ the random future time of the CoCo trigger event, the (random) net present value of the perpetual CoCo under our assumptions (A1)-(A3) is given by

$$P(\tau) := c \int_0^\tau e^{-r\,u}\,\mathrm{d}u + e^{-r\,\tau}\,\alpha\,S_\tau,$$

where $\alpha > 0$ denotes the conversion ratio, i.e. the number of shares per unit bond nominal that is received as a result of the CoCo trigger event. Arbitrage pricing theory suggests that the CoCo price is given as $\mathbb{E}[P(\tau)]$, which necessitates a reasonable stochastic model for the random variable τ. However, and this is a major novelty of the present approach, we do not impose a model for τ, because it is not an easy task, as has been mentioned earlier. Instead, we put ourselves into the shoes of a CoCo investor and make the following conservative assumption:

(A4) **Adverse entity assumption:** The random variable τ is an $(\mathcal{F}_t)$-stopping time which is chosen by an external, "adverse" market entity, so that the CoCo-investor's expected profit is minimized.

The adverse market entity in assumption (A4) may be thought of as either the regulator or the CoCo-issuing bank itself. If one is not willing to accept this assumption, our derivation below does not provide the CoCo price, but instead a lower bound on this price, so that it is still useful. However, it is our intuition that this lower bound is quite sharp, because the timing of the CoCo trigger is most likely to be not in the CoCo-investor's favor. This reflects the core idea of this instrument, acting as a buffer for more secured debt holders in case the issuing bank runs into financial turmoil.
Having collected all assumptions (A1)-(A4), the fair CoCo price is reasonably defined as

$$p := \inf_{\tau \in \mathcal{T}} \big\{ \mathbb{E}[P(\tau)] \big\},$$

where $\mathcal{T}$ denotes the set of all $(\mathcal{F}_t)$-stopping times. The value p can be computed in closed form, which is the content of the following theorem, and the main contribution of the present article.

Theorem 2.2 (Price of perpetual CoCo)
Under assumptions (A1)-(A4) the value of the perpetual CoCo is given by the formula

$$\boxed{p = \begin{cases} \alpha\, S_0, & \text{if } \alpha\, S_0 \leq \frac{2\,c}{\sigma^2+2\,r} \\ \frac{c}{r} - \frac{c\,\sigma^2}{r\,(\sigma^2+2\,r)} \left(\frac{2\,c}{\alpha\, S_0\,(\sigma^2+2\,r)}\right)^{\frac{2\,r}{\sigma^2}}, & \text{else} \end{cases}}$$

and the optimal stopping rule for the adverse entity is given by

$$\boxed{\tau_* := \inf \Big\{ t > 0 \,:\, \alpha\, S_t \leq \frac{2\,c}{\sigma^2 + 2\,r} \Big\}.}$$

Proof
Postponed to Appendix B. ☐

We like to remark at this point that the formula is almost identical to (Black, Cox, 1976, Formula (16), p. 364). The latter formula gives the value $F = F(V_0)$ of a risky perpetual bond in dependence on the current firm value V_0 within the first structural credit risk model with endogenous default as

$$F(V_0) = \frac{C}{r} - \frac{C\,\sigma^2}{r\,(\sigma^2+2\,r)} \left(\frac{2\,C}{V_0\,(\sigma^2+2\,r)}\right)^{\frac{2\,r}{\sigma^2}},$$

where C denotes the absolute amount of coupon that is paid per unit of time[4]. With $C = c$ it is observed that $F(\alpha\, S_0) = p$ in the case $\alpha\, S_0 > 2\,c/(\sigma^2 + 2\,r)$. This similarity is interesting, because the economic interpretation and derivation is different in both cases. The common ground stems from the fact that the two different mathematical derivations involve the optimal choice of a threshold level for a geometric Brownian motion, which represents the stock price in the present paper and a firm value in Black, Cox (1976).

[4]A unit of time must be thought of as one year here, since r is an annual rate.

Remark 2.3 (Implications of Theorem 2.2)
We collect a number of conclusions that can be drawn from Theorem 2.2 in the sequel.

(a) **Structure of the result:** The most adverse trigger timing for the CoCo investor is shown to be the point in time when the stock price first breaches the level $L_* := 2\,c/(\alpha\,(\sigma^2 + 2\,r))$ from above. Intuitively, L_* increases in the coupon rate c, because – thinking of the issuer as being the adverse entity – the higher the coupon to be paid by the issuer the stronger his or her incentive to quit those payments to the investor. Similarly, the trigger level L_* decreases in the stock volatility because, intuitively, higher volatility implies a higher probability that the stock price falls below a low level within short time. Furthermore, the CoCo price is an increasing function in the stock price S_0, increasing to the value c/r of a risk-free perpetual bond as $S_0 \to \infty$. These properties are visualized in Figure 1.

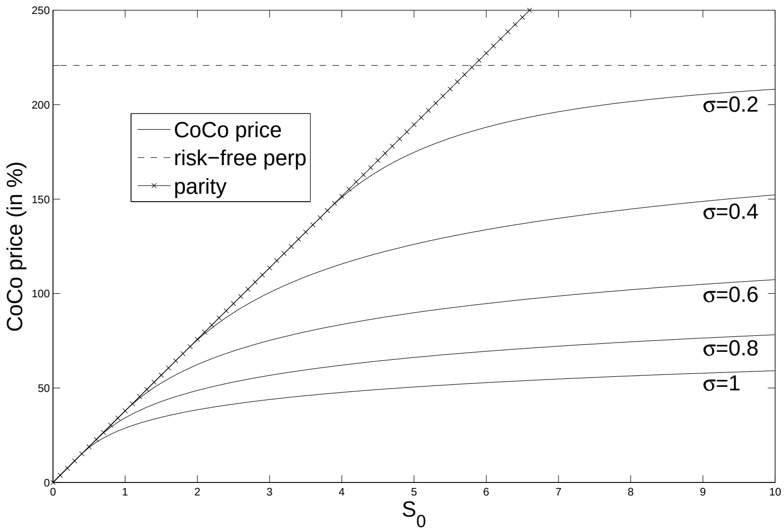

Fig. 1: Visualization of the CoCo pricing formula in dependence of the current stock price S_0, for varying values of the volatility parameter σ. The other parameters are chosen as follows: $r = 3.74\%$ according to Remark 2.3(c), $S_0 = 4.1581$, $c = 8.25\%$, $\alpha = 0.3788$.

(b) **Reasonable trigger level:** The pricing approaches described in De Spiegeleer, Schoutens (2012) rely on the definition of the trigger arrival as the first hitting time of the stock price below a certain trigger level L, see (2). However, L is model input and it is not clear how to choose it appropriately. Given stock volatility σ, interest rate parameter r, and CoCo covenants c and α, Theorem 2.2 spits out the trigger level L_* which is most conservative from an investor's point of view, and therefore serves as a good candidate for putting it into the formulas of De Spiegeleer, Schoutens (2012).

(c) **The choice of r:** Assumption (A1) of a flat interest rate term structure is restrictive. Nevertheless, Theorem 2.2 shows that the CoCo price has the nice structure of being equal to the price of a risk-free perpetual c/r (independently of σ) minus a correction term (depending on σ). Denoting the price of a risk-free perpetual by $p_r := c/r$, the CoCo price may be rewritten in terms of p_r as

$$p = p_r - \Big(p_r - \frac{2}{2\,p_r + \sigma^2/c}\Big)\,\Big(\frac{\alpha\, S_0\,(2\,p_r + \sigma^2/c)}{2}\Big)^{-\frac{2\,c}{p_r\,\sigma^2}}. \tag{3}$$

This provides an approximate formula for the CoCo price without the short rate parameter r, in which the (elsewhere computed) value of a risk-free perpetual may be plugged in. For instance, having bootstrapped a determinsitic short rate curve $t \mapsto r(t)$ from observed swap rates according to common market standard, e.g. along one of the methods described in Hagan, West (2006), one might plug the value $p_r := c\int_0^\infty e^{-r(t)}\,\mathrm{d}t$ into (3). This is equivalent to the choice $r := 1/\int_0^\infty e^{-r(t)}\,\mathrm{d}t$ and at least implies consistency of the first term in the formula with the pricing of risk-free debt within one's pricing library. In this way, the error induced by assumption (A1) is attributed completely to the correction term, which might be fine-tuned separately by different means (yet to be researched).

(d) **The trigger probability:** The probability distribution of the trigger arrival time is known in closed form, see, e.g., Black, Cox (1976). Provided $S_0 > L_*$ (otherwise $\tau_* = 0$ almost surely), it is given by

$$\begin{aligned}\mathbb{P}(\tau_* \leq t) = 1 - \Phi\left(\frac{\left(r - \frac{\sigma^2}{2}\right)t - \log\left(\frac{L_*}{S_0}\right)}{\sigma\sqrt{t}}\right)\\ + e^{\left(\frac{2\,r}{\sigma^2}-1\right)\,\log\left(\frac{L_*}{S_0}\right)}\,\Phi\left(\frac{\left(r - \frac{\sigma^2}{2}\right)t + \log\left(\frac{L_*}{S_0}\right)}{\sigma\sqrt{t}}\right),\end{aligned}$$

where Φ denotes the distribution function of a standard normally distributed random variable. The probability that the trigger event occurs not before time t is decreasing in the volatility parameter, see Figure 2. The density of τ_* for $S_0 > L_*$ is given by

$$f_{\tau_*}(t) = \frac{\log\left(\frac{S_0}{L_*}\right)}{t\,\sigma\,\sqrt{2\,\pi\,t}}\,\exp\Big(-\frac{\left(\log\left(\frac{S_0}{L_*}\right) + \left(r - \frac{\sigma^2}{2}\right)t\right)^2}{2\,\sigma^2\,t}\Big).$$

(e) **Delta-hedge ratio:** CoCo investors might be interested in the computation of a hedge ratio for the CoCo price with respect to the underlying stock price. To this end, the closed formula of Theorem 2.2 readily implies

$$\frac{\partial}{\partial S_0}p = \begin{cases}\alpha & \text{, if } \alpha\, S_0 \leq \frac{2\,c}{\sigma^2+2\,r}\\ \alpha^{-\frac{-2\,r}{\sigma^2}}\,\left(\frac{2\,c}{S_0\,(\sigma^2+2\,r)}\right)^{\frac{2\,r+\sigma^2}{\sigma^2}} & \text{, else}\end{cases}.$$

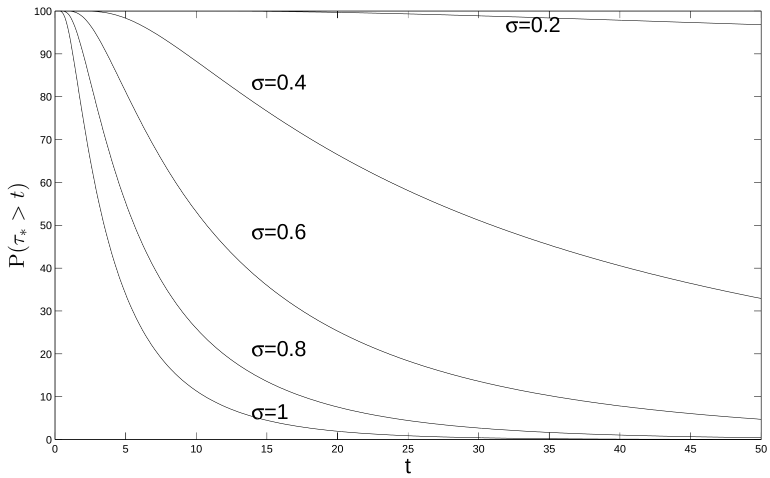

Fig. 2: Visualization of the probability $t \mapsto \mathbb{P}(\tau_* > t)$ that τ_* does not occur before time t for varying volatility parameters σ and the same specifications as in Figure 1.

3 Perpetual CoCos with an additional call right

Notice that all currently outstanding CoCos provide their issuing bank with additional call rights at par, starting at some future point in time $T > 0$, i.e. the issuing bank is allowed to repay the debt early. Moreover, at T, if the CoCo is not called, there is a step up feature, i.e. the coupon rate is slightly higher than before T. For the sake of simplicity, we assume that the issuer has precisely one call right at T. In reality the call right is continuous (or Bermudan style) starting from T. We comment on the possibility of further generalization in Remark 3.1 below. Denoting the coupon rate before T by c_1 and after T by c_2, this results in a net present value of

$$\tilde{P}(\tau) := \Big(c_1 \int_0^{\tau} e^{-r\,u}\,\mathrm{d}u + e^{-r\,\tau}\,\alpha\,S_\tau\Big)\,1_{\{\tau \leq T\}}$$
$$+ \Big(c_1 \int_0^{T} e^{-r\,u}\,\mathrm{d}u + \min\Big\{e^{-r\,T}, c_2 \int_T^{\tau} e^{-r\,u}\,\mathrm{d}u + e^{-r\,\tau}\,\alpha\,S_\tau\Big\}\Big)\,1_{\{\tau > T\}}.$$

This complicates our problem significantly, because the callable perpetual CoCo price is now reasonably defined as

$$\tilde{p} := \inf_{\tau \in \mathcal{T}} \{\mathbb{E}[\tilde{P}(\tau)]\}$$

and the respective minimization is more difficult. In economic terms, the adverse entity may not only opt for conversion in case of a falling stock price but can also stop his losses at T in case the stock price at T is so high that he or she thinks it is better to call than hope for a quick decline of the stock. We can make use of the knowledge about the optimal solution in Theorem 2.2 in order to tackle this problem as well. To this end, we emphasize the dependence of p in Theorem 2.2 on the coupon rate c explicitly:

$$p^{(c)}(S_0) := \begin{cases} \alpha\,S_0, & \text{if } \alpha\,S_0 \leq \frac{2\,c}{\sigma^2+2\,r} \\ \frac{c}{r} - \frac{c\,\sigma^2}{r\,(\sigma^2+2\,r)}\Big(\frac{2\,c}{\alpha\,S_0\,(\sigma^2+2\,r)}\Big)^{\frac{2\,r}{\sigma^2}}, & \text{else} \end{cases}.$$

With the tower property of conditional expectation and the Markov property of Brownian motion, it follows that

$$\tilde{p} = \inf_{\tau \in \mathcal{T}} \{\mathbb{E}[\hat{P}(\tau)]\}, \tag{4}$$

with

$$\hat{P}(\tau) := \begin{cases} c_1 \int_0^\tau e^{-r\,u}\,\mathrm{d}u + e^{-r\,\tau}\,\alpha\,S_\tau & \text{, if } \tau \leq T \\ c_1 \int_0^T e^{-r\,u}\,\mathrm{d}u + e^{-r\,T}\,\min\{1, p^{(c_2)}(S_T)\} & \text{, else} \end{cases}.$$

In particular, the value of $\hat{P}(\tau)$ in the case $\tau > T$ is independent of τ. In economic terms, the adverse entity may either opt for conversion into equity before time T, or wait until time T and then decide between a call at par and the payment of a non-callable perpetual CoCo with coupon rate c_2. The remaining minimization problem is now straightforward to implement using tree pricing techniques. Required is only an efficient trinomial tree implementation for $\{S_t\}_{t \in [0,T]}$, which is a standard exercise.

Remark 3.1 (Potential for generalizations)
The use of tree pricing for the solution of (4) has the advantage that it allows to relax assumption (A1) to the assumption that the short rate needs not be constant but may be a deterministic function $t \mapsto r(t)$ that becomes constant only after the call date T, which is a significant improvement for practical purposes. Moreover, the inclusion of more call rights in a period after T, say $[T, T+S]$, is conceptually straightforward. One has to define the value at the end note of the tree, corresponding to $T+S$, like above using the solution of Theorem 2.2, and within the tree pricing backwardation procedure include additional call decisions at each note between T and $T+S$. Hence, for the sake of clarity of presentation we consider it sufficient to illustrate the case of only one call right.

3.1 Quick numerical approximation

We provide an alternative ansatz to compute the price $\tilde{p}$ approximately without tree pricing in the sequel. The structure of the minimization problem suggests to make the following presumption.

(A5) **Presumption on the optimal stopping before T:** The optimal stopping time for the problem (4) is of the form τ_L as in (2) for some optimally chosen threshold L.

Under assumption (A5) the expectation $g_L(S_0) := \mathbb{E}[\hat{P}(\tau_L)]$ may be minimized in L in order to yield $\tilde{p}$. Even if assumption (A5) does not hold true, we obtain a very good approximation for $\tilde{p}$, i.e. we work with the approximation

$$\tilde{p} = \inf_{\tau \in \mathcal{T}} \{\mathbb{E}[\hat{P}(\tau)]\} \approx \inf_{L>0} \{\mathbb{E}[\hat{P}(\tau_L)]\} = \inf_{L>0} \{g_L(S_0)\}.$$

Our numerical experiments suggest that assumption (A5) is indeed not too bad, see, e.g., Figure 3. The following lemma provides $g_L(S_0)$ in numerically convenient form.

Lemma 3.2 (Computation of required expectation)
For $x > 0$ the value $g_L(x)$ is given as

$$
\begin{aligned}
g_L(x) = &\int_0^T \Big(\frac{c_1}{r}\Big(1 - e^{-r\,u}\Big) + \alpha\,L\,e^{-r\,u}\Big)\,\frac{\log\Big(\frac{x}{L}\Big)}{u\,\sigma\,\sqrt{2\,\pi\,u}} \times \\
&\quad \times \exp\Bigg(-\frac{\Big(\log\Big(\frac{x}{L} + \Big(r - \frac{\sigma^2}{2}\Big)\,u\Big)\Big)^2}{2\,\sigma^2\,u}\Bigg)\,\mathrm{d}u \\
&+ \frac{c_1}{r}\Big(1 - e^{-r\,T}\Big)\Bigg(\Phi\Bigg(\frac{\Big(r - \frac{\sigma^2}{2}\Big)\,T - \log\Big(\frac{L}{x}\Big)}{\sigma\,\sqrt{T}}\Bigg) \\
&\quad - e^{\Big(\frac{2\,r}{\sigma^2} - 1\Big)\,\log\Big(\frac{L}{x}\Big)}\,\Phi\Bigg(\frac{\Big(r - \frac{\sigma^2}{2}\Big)\,T + \log\Big(\frac{L}{x}\Big)}{\sigma\,\sqrt{T}}\Bigg)\Bigg) \\
&+ e^{-r\,T}\int_{-\infty}^{\frac{1}{\sigma}\log\big(\frac{x}{L}\big)}\Bigg(1 - \exp\Bigg(-\frac{2\,\log\Big(\frac{x}{L}\Big)}{\sigma\,T}\,\Big(\frac{1}{\sigma}\,\log\Big(\frac{x}{L}\Big) - u\Big)\Bigg)\Bigg) \times \\
&\quad \times \min\Big\{1, p^{(c_2)}\Big(x\,e^{-\sigma\,u}\Big)\Big\}\,\frac{1}{\sqrt{2\,\pi\,T}}\,e^{-\frac{\Big(u - \Big(\frac{\sigma}{2} - \frac{r}{\sigma}\Big)\,T\Big)^2}{2\,T}}\,\mathrm{d}u
\end{aligned}
$$

Proof
Postponed to Appendix C. □

Akin to the proof of Theorem 2.2, numerical experiments confirm our intuition that $g_L(x) \geq g_{L_*}(x)$ uniformly in x, where L_* is chosen to be the unique threshold L satisfying $\frac{\partial}{\partial x} g_L(L) = \alpha$. The latter equation can be solved for L_* within seconds via a bisection routine. Consequently, the approximation $\tilde{p} \approx g_{L^*}$ is a reasonable and efficiently computable formula for the perpetual CoCo with additional call right. Figure 3 visualizes this approximation.
Finally, let us remark that the call probability of the CoCo at time T may be computed in closed form. The CoCo is called if and only if it is not converted into equity before T and $p^{(c_2)}(S_T) > 1$. Denoting by L_* the unique solution of the equation $\frac{\partial}{\partial x} g_L(L) = \alpha$ for L, this probability is computed as

$$
\begin{aligned}
&\mathbb{P}\Big(\min_{t\,:\,t \leq T}\{S_t\} > L_*,\, p^{(c_2)}(S_T) > 1\Big) \\
&\quad = \mathbb{P}\Big(\max_{t\,:\,t \leq T}\{\hat{W}_t\} < \frac{1}{\sigma}\,\log\Big(\frac{S_0}{L_*}\Big),\, p^{(c_2)}\Big(S_0\,e^{-\sigma\,\hat{W}_T}\Big) > 1\Big) \\
&\quad = \int_{-\infty}^{\frac{1}{\sigma}\log\Big(\frac{S_0}{(p^{(c_2)})^{-1}(1)}\Big)} \Bigg(1 - e^{-\max\Big\{\frac{2\,\log\big(\frac{S_0}{L_*}\big)}{\sigma\,T}\,\Big(\frac{1}{\sigma}\,\log\Big(\frac{S_0}{L_*}\Big) - w\Big), 0\Big\}}\Bigg) \times \\
&\qquad\qquad \times \frac{1}{\sqrt{2\,\pi\,T}}\,e^{-\frac{\Big(w - \Big(\frac{\sigma}{2} - \frac{r}{\sigma}\Big)\,T\Big)^2}{2\,T}}\,\mathrm{d}w,
\end{aligned}
$$

where knowledge about the conditional distribution of the running maximum of $\hat{W}_t := (\sigma/2 - r/\sigma)\,t + W_t$ until T given $\hat{W}_T$ was used, see, e.g., (Shreve, 2004, Theorem 7.2.1, p. 296). The

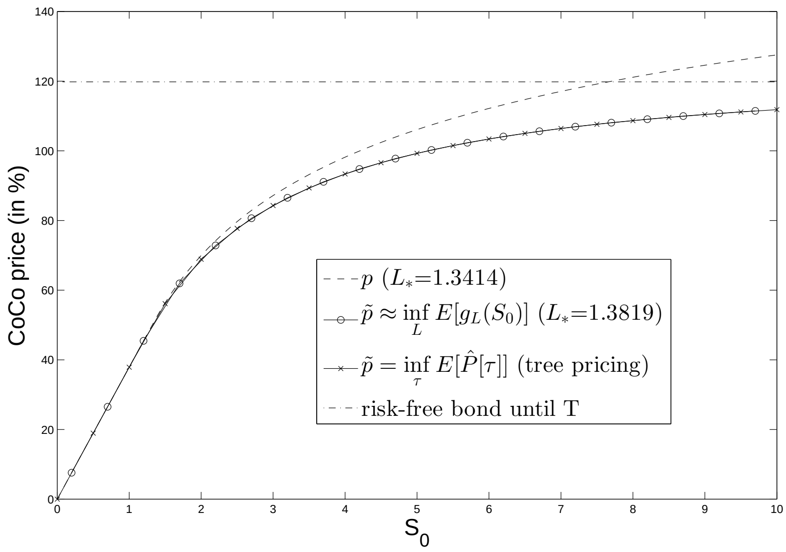

Fig. 3: Visualization of the CoCo price $\tilde{p}$ and the approximation formula $\tilde{p} \approx g_{L_*}$ in dependence of the current stock price S_0, for $\sigma = 50\%$. The other parameters are chosen like in Figure 1, except for the coupons which are now $c_1 = 0.0825$ and $c_2 = 0.06075 + r$ (step up coupon). The optimal threshold in the minimization routine (4) under assumption (A5) is $L_* = 1.3847$, whereas the optimal threshold according to Theorem 2.2, when ignoring the call right and step up coupon (assuming $c = c_1$), is $L_* = 1.3414$. The plot also shows the value $c_1 \int_0^T e^{-r\,t}\,\mathrm{d}t + e^{-r\,T}$ of the risk-free bond with coupon c_1 and maturity T, which acts as an upper bound for $\tilde{p}$.

required value $(p^{(c_2)})^{-1}(1)$ is computed in closed form as

$$(p^{(c_2)})^{-1}(1) = \begin{cases} 1/\alpha & \text{, if } \frac{2\,c_2}{\sigma^2+2\,r} > 1 \\ \frac{2}{\alpha}\left(\frac{\sigma^2}{c_2 - r}\right)^{\frac{\sigma^2}{2\,r}} \left(\frac{c_2}{\sigma^2+2\,r}\right)^{\frac{\sigma^2+2\,r}{2\,r}} & \text{, else} \end{cases}.$$

This probability is visualized in Figure 4 for the CoCo example of Figure 3. It is observed how this probability approaches one for $S_0 \to \infty$.

4 A real-world example

We consider the CoCo BACR 9.25 GBP Perp, which was recently issued by Barclays PLC briefly before the swiss bank Credit Suisse was taken over by its peer UBS and Credit Suisse AT1 CoCos have been written down to zero. The latter event had an enormous impact on the CoCo market, sending the average CoCo prices down 10-20$\%$, since the market feared more CoCos could be written down or converted into equity. We only consider this particular CoCo as one representative for the general CoCo market, similar findings hold for other CoCos as well. The BACR 9.25 GBP Perp CoCo has the following parameters according to the notation in this article: $c_1 = 9.25\%$, $c_2 =$ 5-year mid-swap rate GBP + 5.639%, $\alpha = 1/1.65$. On 30 March 2023 the equity equals around 1.4558 GBP and we choose $r = 3.11\%$ as a proxy for the flat interest rate required in our pricing routine. This number is computed according to the logic described in Remark

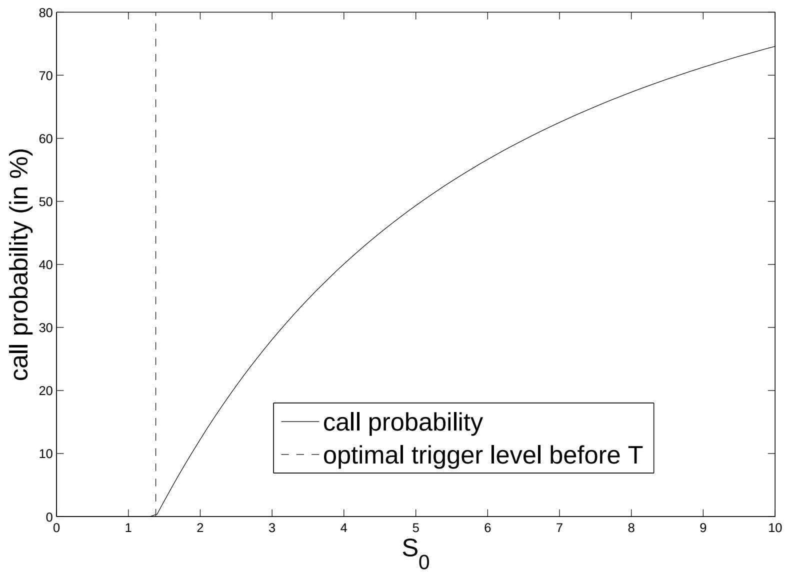

Fig. 4: Visualization of the probability that the CoCo of Figure 3 is called at T_1 in dependence on S_0. Clearly, if the current stock price is already below L_* the call probability is zero because conversion into equity is imminent.

2.3(c). The CoCo price is around 87.5%, which corresponds to a parameter $\sigma = 32.11\%$ according to our methodology. It is remarkable that this is almost exactly the value of the historical stock volatility over the past 260 business days on 30 March 2023. Consequently, our method provides an almost accurate explanation for the current market price. However, at the beginning of the month the CoCo traded around 100%, where it was issued, and this price was absolutely not explainable with the presented technique. In intuitive terms, before the Credit Suisse incident, our adverse entity assumption was way too conservative to explain observed market prices. However, after the recent market drop our conservative technique is suddenly capable of explaining the prices. Consequently, our method might be a suitable tool to explain CoCo prices in a distressed market environment.

5 Conclusion and Outlook

Within a Black-Scholes setup we managed to derive a closed formula for perpetual CoCos with full equity conversion feature, which is applicable to one fifth of all currently outstanding CoCos. We furthermore discussed some applications and implications of this result.

Concerning an outlook of potential further research, relaxations of the assumptions (A1)-(A3) are of high relevance in practice. For instance, a relaxation of assumption (A3) to regime-switching stock price processes might be achievable via similar techniques as in Guo, Zhang (2004). Regarding a relaxation of assumption (A2), we have already provided one feasible solution in Section 3, which sheds some light on how the required techniques change with further relaxations (e.g. to multiple calls). Discretizing the coupon payments or relaxing assumption (A1) to a non-constant but deterministic short rate leads to similar difficulties, which might be overcome in future research at the cost of a more involved expression for f_L, resp. g_L.

Appendix A: capital structure

A bank's capital structure looks roughly as follows under Basel III (seniority increasing), in accordance with the Capital Requirement Regulation (CRR) and the Capital Requirement Directive IV (CRD-IV). RWA stands for *risk-weighted assets*, a notion that is defined in the Basel III legislation.

(1) **Tier 1 Capital (T1):**

(1.1) **Common Equity Tier 1 (CET1):**

- Must comprise $\geq 4.5\%$ of RWA.
- Basically consists of regular shares, preferred stock, retained earnings.

(1.2) **Additional Tier 1 (AT1):**

- Together with CET1 must comprise $\geq 6\%$ of RWA, must not exceed 1.5% of RWA.
- Going concern capital, i.e. capital intended to sustain the company, subordinate to Tier 2.
- Perpetual bonds with right to omit coupons under certain circumstances (no calls along with significant step-ups as incentive to call), certain preference shares.

(2) **Tier 2 Capital (T2):**

- Together with T1 must comprise $\geq 8\%$ of RWA.
- Gone concern capital, i.e. capital acting as buffer in case of bankruptcy but not to sustain the company in case of no bankruptcy, subordinate to senior debt.
- Regular bonds, subordinate to senior debt.

(3) **Senior Debt Capital:**

- Regular bonds, senior to Tier 1 and Tier 2.

The indicated minimal capital requirements are strict, i.e. the bank is closed if it does not meet them. The CRR demands in addition that the bank holds a *combined capital buffer* comprised of CET1. Adding this buffer to the minimal requirements, legislation demands that CET1 is at least 7% of RWA (slightly depending on the national buffer requirements). What is the use of this combined buffer? Banks that fail to meet the combined buffer requirement must calculate a *maximum distributable amount (MDA)*, which serves as basis for the payment of dividends on the bank's stocks or bonuses to its employees. The *CET1 ratio*, which is often involved in the definition of CoCo trigger events, is computed as CET1 divided by the sum of RWA. Notice that the definition of RWA is hence crucial for the computation of the CET1 ratio, and hence for the definition of the CoCo trigger event. Consequently, a regulatory change of the computation method for RWA may alter the definition of the CoCo trigger event and might lead to a regulatory call of the CoCo. Such a scenario is not too unlikely and must in particular be kept in mind by investors into CoCos that trade significantly above par.

Appendix B: proof of Theorem 2.2

Our strategy is to rewrite the problem in such a way that it resembles a solved one. To this end, it is useful to switch from $\inf$ to $\sup$ and noticing that

$$-p = \sup_{\tau\in\mathcal{T}} \{\mathbb{E}[-P(\tau)]\},$$
$$-P(\tau) = -\frac{c}{r}\left(1-e^{-r\,\tau}\right) - \alpha\, e^{-r\,\tau}\, S_\tau.$$

This maximization problem resembles the computation of the perpetual put option, see, e.g., Gerber, Shiu (1994). Indeed, the known derivation of the perpetual put option problem – we work along the lines of (Shreve, 2004, p. 345ff) – can be enhanced in order to solve the present optimization. In detail, the proof relies on the following auxiliary steps.

(I) For $L>0$ denote $\tau_L := \inf\{t>0 \,:\, S_t \le L\}$ as in (2) and $f_L(S_0) := \mathbb{E}[-P(\tau_L)]$. It then follows that

$$f_L(S_0) = \begin{cases} -\alpha\, S_0, & \text{if } S_0 \le L \\ -\frac{c}{r} + \left(\frac{c}{r} - \alpha\, L\right)\left(\frac{S_0}{L}\right)^{-\frac{2\,r}{\sigma^2}}, & \text{else} \end{cases}.$$

Proof
If $S_0 \le L$, then $\tau_L = 0$ and $f_L(S_0) = -\alpha\, S_0$, as claimed. If $S_0 > L$, we obtain

$$\begin{aligned} f_L(S_0) &= \mathbb{E}\Big[-\frac{c}{r}\left(1-e^{-r\,\tau_L}\right) - \alpha\, e^{-r\,\tau_L}\, L\Big] \\ &= -\frac{c}{r} + \left(\frac{c}{r} - \alpha\, L\right)\mathbb{E}\Big[e^{-r\,\tau_L}\Big] \\ &= -\frac{c}{r} + \left(\frac{c}{r} - \alpha\, L\right)\left(\frac{S_0}{L}\right)^{-\frac{2\,r}{\sigma^2}}, \end{aligned}$$

where the last equality follows like in the proof of (Shreve, 2004, Lemma 8.3.4, p. 348–349). $\square$

(II) We have $f_L(x) \le f_{L_*}(x)$ uniformly in $x>0$ for

$$L_* := \frac{2\,c}{\alpha\,(\sigma^2 + 2\,r)}.$$

Proof
On the interval $[L,\infty)$ we have seen in (1) that

$$f_L(x) = -\frac{c}{r} + \alpha\left(\frac{c}{r\,\alpha} - L\right)\left(\frac{x}{L}\right)^{-\frac{2\,r}{\sigma^2}} := \hat{f}_L(x).$$

In order to maximize $\hat{f}_L(x)$ uniformly in x it suffices to maximize the function $g \,:\, L \mapsto (c/(r\,\alpha) - L)\, L^{2\,r/\sigma^2}$ in L. It is not difficult to observe that

$$L_* := \arg\max_{L>0}\{g(L)\} = \frac{2\,c}{\alpha\,(\sigma^2+2\,r)}.$$

Now for $x>0$ arbitrary we have

$$\begin{aligned} \sup_{L\ge 0}\{f_L(x)\} &= \max\Big\{\sup_{L\,:\,L>x}\{f_L(x)\},\ \sup_{L\,:\,L\le x}\{f_L(x)\}\Big\} \\ &= \max\Big\{-\alpha\, x,\ \sup_{L\,:\,L\le x}\{\hat{f}_L(x)\}\Big\}. \end{aligned}$$

Since $\hat{f}_{L_*}(L_*) = -\alpha\, L_*$ and $\hat{f}'_{L_*}(x) \geq (-\alpha\, x)' = -\alpha$ for $x > L_*$, we observe

$$\hat{f}_{L_*}(x) \geq -\alpha\, x \text{ for } x \geq L_*, \quad f_{L_*}(x) \geq -\alpha\, x \text{ for } x > 0. \quad (5)$$

Now consider $x \geq L_*$. Then

$$\begin{aligned}\sup_{L\geq 0}\{f_L(x)\} &= \max\big\{-\alpha\, x,\ \sup_{L:\,L\leq x}\{\hat{f}_L(x)\}\big\}\\ &= \max\big\{-\alpha\, x,\ \hat{f}_{L_*}(x)\}\big\}\\ &= \hat{f}_{L_*}(x) = f_{L_*}(x),\end{aligned}$$

where (5) is used in the third equality. For $x < L_*$ we see

$$\begin{aligned}\sup_{L\geq 0}\{f_L(x)\} &= \max\big\{-\alpha\, x,\ \sup_{L:\,L\leq x}\{\hat{f}_L(x)\}\big\}\\ &= \max\big\{-\alpha\, x,\ \sup_{L:\,L\leq x}\{-\frac{c}{r} + \alpha\, x^{-\frac{2\,r}{\sigma^2}}\, g(L)\}\big\}\\ &= \max\big\{-\alpha\, x,\ -\frac{c}{r} + \alpha\, x^{-\frac{2\,r}{\sigma^2}}\, g(x)\big\}\\ &= -\alpha\, x = f_{L_*}(x).\end{aligned}$$

□

(III) The stochastic process

$$t \mapsto -\frac{c}{r}\Big(1 - e^{-r\,t}\Big) + e^{-r\,t}\, f_{L_*}(S_t), \quad t \geq 0,$$

is a super-martingale.

Proof
An application of Itô's formula[5] implies that the drift term of the process in the statement is given by

$$e^{-r\,t}\Big(-c - r\, f_{L_*}(S_t) + r\, S_t\, f'_{L_*}(S_t) + \frac{\sigma^2}{2}\, S_t^2\, f''_{L_*}(S_t)\Big)\,\mathrm{d}t.$$

Plugging in the respective formulas for the first and second derivative of f_{L_*}, it follows that the drift equals $-c\, e^{-r\,t} \leq 0$ whenever $S_t \leq L_*$ and equals zero whenever $S_t > L_*$. Hence, the drift is always non-positive and the process is a super-martingale, as claimed. □

Using (I)-(III) above, the claim can now be established. Let $\tau \in \mathcal{T}$ arbitrary. Then, applying (III) in the first, Fatou's lemma in the second, and (5) in the third inequality below,

$$\begin{aligned}f_{L_*}(S_0) &= \lim_{n\to\infty}\Big\{-\frac{c}{r}\Big(1 - e^{-r\cdot 0}\Big) + e^{-r\cdot 0}\, f_{L_*}(S_0)\Big\}\\ &\geq \lim_{n\to\infty}\mathbb{E}\Big[-\frac{c}{r}\Big(1 - e^{-r\,\min\{\tau,n\}}\Big) + e^{-r\,\min\{\tau,n\}}\, f_{L_*}(S_{\min\{\tau,n\}})\Big]\\ &\geq \mathbb{E}\Big[-\frac{c}{r}\Big(1 - e^{-r\,\tau}\Big) + e^{-r\,\tau}\, f_{L_*}(S_\tau)\Big]\\ &\geq \mathbb{E}\Big[-\frac{c}{r}\Big(1 - e^{-r\,\tau}\Big) - e^{-r\,\tau}\,\alpha\, S_\tau\Big] = \mathbb{E}[-P(\tau)],\end{aligned}$$

yielding $f_{L_*}(S_0) \geq -p$, from which the claim follows.

[5] More formally, Itô-Tanaka's formula, because f''_{L_*} has a discontinuity at L_*, see, e.g., (Mörters, Peres, 2010, Remark 7.33, p. 210) for details.

Appendix C: Proof of Lemma 3.2

We observe

$$\mathbb{E}[\hat{P}(\tau_L)] = \int_0^T \Big(\frac{c_1}{r}\Big(1-e^{-r\,t}\Big) + e^{-r\,t}\,\alpha\,L\Big)\,\mathrm{d}\mathbb{P}(\tau_L \leq t)$$
$$+\frac{c_1}{r}\Big(1-e^{-r\,T}\Big)\,\mathbb{P}(\tau_L > T)$$
$$+\,e^{-r\,T}\,\underbrace{\mathbb{E}\Big[1_{\{\min\limits_{t\,:\,t\leq T}\{S_t\}>L\}}\,\min\{1,-f_{L_*}(S_T)\}\Big]}_{(*)}.$$

Furthermore, we apply the fact that $\{W_t\} \stackrel{d}{=} \{-W_t\}$ and denote $\hat{W}_t := (-r/\sigma+\sigma/2)\,t+W_t$, enabling us to rewrite the remaining expectation value as

$$(*) = \mathbb{E}\Big[1_{\Big\{\max\limits_{t\,:\,t\leq T}\{\hat{W}_t\}<\frac{1}{\sigma}\,\log\Big(\frac{S_0}{L}\Big)\Big\}}\,\min\{1,-f_{L_*}(S_0\,e^{-\sigma\,\hat{W}_T})\}\Big]$$
$$= \mathbb{E}\Big[\min\{1,-f_{L_*}(S_0\,e^{-\sigma\,\hat{W}_T})\}\times$$
$$\times\,\mathbb{E}\Big[\int_{\hat{W}_T}^{\max\left\{\hat{W}_T,\frac{1}{\sigma}\,\log\left(\frac{S_0}{L}\right)\right\}} \frac{2\,(2\,m-\hat{W}_T)}{T}\,e^{-\frac{2\,m}{T}\,(m-\hat{W}_T)}\,\mathrm{d}m\,\Big|\,\hat{W}_T\Big]\Big]$$
$$= \mathbb{E}\Big[\min\{1,-f_{L_*}(S_0\,e^{-\sigma\,\hat{W}_T})\}\times$$
$$\times\,\Big(1-\exp\Big(-\frac{2\,\log\Big(\frac{S_0}{L}\Big)}{\sigma\,T}\,\max\Big\{\frac{1}{\sigma}\,\log\Big(\frac{S_0}{L}\Big)-\hat{W}_T,0\Big\}\Big)\Big)\Big],$$

where we applied knowledge about the conditional density of the running maximum of $\{\hat{W}_t\}$ until T given $\hat{W}_T$, see, e.g., (Shreve, 2004, Theorem 7.2.1, p. 296). Consequently, the expected value $(*)$ may be computed as a one-dimensional integral over closed-form expressions with respect to the normal density of $\hat{W}_T$. Recalling the closed-form solution of the density and distribution function of τ_L (replace L_* by L in the respective formulas of Remark 2.3(d)), the claimed formula for $g_L(S_0)$ is obtained.

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References

B. Albul, D.M. Jaffee, A. Tchistyi, Contingent capital debt and capital structure decisions, *Working Paper, University of Berkeley* (2010).

F. Black, J. Cox, Valuing corporate securities: some effects of bond indenture provisions, *Journal of Finance* **31** (1976) pp. 351–367.

D. Brigo, J. Garcia, N. Pede, CoCo bonds valuation with equity- and credit-calibrated first passsage structural models, *Working paper* (2013).

P. Cheridito, Z. Xu, Pricing and hedging CoCos, *Working paper* (2013).

J. De Spiegeleer, W. Schoutens, Pricing Contingent Convertibles: a derivatives approach, *Journal of Derivatives* **20** (2012) pp. 27–36.

H.U. Gerber, E.S.W. Shiu, Martingale approach to pricing perpetual American options, *ASTIN Bulletin* **24** (1994) pp. 195–220.

P. Glasserman, B. Nouri, Contingent capital with a capital-ratio trigger, *Management Science* **58:10** (2012) pp. 1816–1833.

J. Glionna, C. Hajiloizou, Y. di Mambro, D. Winnicki, S. Gupta, The CoCo Handbook Vol. 3, *Barclays Credit Research*, March 2014.

X. Guo, Q. Zhang, Closed-form solutions for perpetual American put options with regime switching, *SIAM Journal of Applied Mathematics* **64** (2004) pp. 2034–2049.

P.S. Hagan, G. West, Interpolation methods for curve construction, *Applied Mathematical Finance* **13:2** (2006) pp. 89–129.

D. Madan, W. Schoutens, Conic coconuts: the pricing of contingent capital notes using conic finance, *Mathematics and Financial Economics* **4** (2011b) pp. 87–106.

D. Madan, W. Schoutens, Conic financial markets and corporate finance, *International Journal of Theoretical and Applied Finance* **14** (2011a) pp. 587–610.

H.E. Leland, Corporate debt value, bond covenants and optimal capital structure, *Journal of Finance* **49** (1994) pp. 1213–1252.

P. Mörters, Y. Peres, Brownian motion, *Cambridge University Press* (2010).

G. Penacchi, A structural model of contingent bank capital, *Working Paper, University of Illinois* (2010).

M. Rubinstein, E. Reiner, Unscrambling the binary code, *Risk Magazine* **4** (1991) pp. 75–83.

S.E. Shreve, Stochastic calculus for finance II: continuous-time models, *Springer* (2004).

S. Sundaresan, Z. Wang, Design of contingent capital with a stock price trigger for mandatory conversion, *Working Paper, Columbia University* (2010).