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CORRIDOR VARIANCE SWAPS

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Abstract

Corridor variance swaps are financial derivative contracts that generalize variance swaps and provide exposure to equity volatility. Whereas they are less liquid and only traded in over-the-counter markets, their payoffs can be replicated by trading strategies that rely only on liquidly traded vanilla options and forward contracts, at least approximately. As a result, the implementation of corridor variance swaps via vanilla options is used in quantitative trading strategies that seek to deliver sophisticated exposure to equity volatility. The present survey introduces corridor variance swaps and explains the aforementioned replication via liquid derivative contracts.

1 Corridor variance swaps

We denote by $S = \{S_t\}_{t \geq 0}$ the value process of some stock, or stock index. With a risk-free and deterministic short-rate assumption $r = \{r(t)\}_{t \geq 0}$ used for discounting future cashflows, and with a constant dividend yield $\delta \geq 0$, we assume that S has the dynamics

$$\mathrm{d}S_t = S_t \, \big((r(t) - \delta) \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t \big)$$

under some risk-neutral pricing measure $\mathbb{Q}$. Here, $W = \{W_t\}_{t \geq 0}$ denotes a standard Brownian motion and $\{\sigma_t\}_{t \geq 0}$ can be any non-negative stochastic volatility process, which is assumed to be observable by market participants, i.e. is adapted to some market filtration $\{\mathcal{F}_t\}_{t \geq 0}$ that contains the natural filtration of W. We furthermore introduce the notation $F_{t,T} := \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t] = S_t \, e^{\int_t^T r(s) - \delta \, \mathrm{d}s}$, $t \in [0, T]$, for the equity forward with maturity T, and we note that

$$\mathrm{d}F_{t,T} = F_{t,T} \, \sigma_t \, \mathrm{d}W_t$$

under $\mathbb{Q}$, i.e. in comparison to S the forward is always a local martingale and has no drift.

A *corridor variance swap* with corridor $[a, b]$ for $0 \leq a < b \leq \infty$ is a financial contract with maturity T and underlying $X = \{X_t\}_{t \in [0,T]}$, where either $X_t = S_t$ or $X_t = F_{t,T}$, which costs zero at inception and delivers its holder at maturity T a payoff depending on the path of X, given by

$$\frac{1}{T} \sum_{i=1}^{n} 1_{(a,b)}(X_{t_{i-1}^{(n)}}) \Big\{ \Big(\frac{X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}}}{X_{t_{i-1}^{(n)}}} \Big)^2 - \bar{\sigma}^2_{CVS} \Big\},$$

where $\bar{\sigma}^2_{CVS}$ is a constant, called the *implied corridor variance* with corridor $[a, b]$ of the underlying X in the period $[0, T]$. Here, $n \in \mathbb{N}$ is a number of monitoring time points, usually the number of business days in $[0, T]$, and the partition $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T$ specifies contractually specified monitoring time points $t_i^{(n)}$. The expression

$$\frac{1}{T}\sum_{i=1}^{n} 1_{(a,b)}(X_{t_{i-1}^{(n)}})\Big(\frac{X_{t_i^{(n)}}-X_{t_{i-1}^{(n)}}}{X_{t_{i-1}^{(n)}}}\Big)^2, \tag{1}$$

is called the *realized corridor variance* in the corridor $[a,b]$ of the underlying X in the period $[0,T]$. In words, a corridor variance swap swaps the realized variance leg into the implied variance leg, where the implied variance must be viewed as the market's expectation/price for the realized variance. The usual *variance swap* arises as a special case for $a=0$ and $b=\infty$. If we let $n\to\infty$ and assume that the mesh $\max_{1\le i\le n}|t_i^{(n)}-t_{i-1}^{(n)}|$ of the partition tends to zero, then the realized corridor variance converges (in probability) to

$$\begin{aligned}\frac{1}{T}\lim_{n\to\infty}\sum_{i=1}^{n} 1_{(a,b)}(X_{t_{i-1}^{(n)}})\Big(\frac{X_{t_i^{(n)}}-X_{t_{i-1}^{(n)}}}{X_{t_{i-1}^{(n)}}}\Big)^2\\ =\frac{1}{T}\int_0^T\frac{1_{(a,b)}(X_t)}{X_t^2}\,\mathrm{d}[X,X]_t=\frac{1}{T}\int_0^T 1_{(a,b)}(X_t)\,\sigma_t^2\,\mathrm{d}t,\end{aligned}$$

where $[X,X]$ denotes the quadratic covariation process of X, which by our assumption of a continuous diffusion process is known to be given by $[X,X]_t=\int_0^t\sigma_u^2\,X_u^2\,\mathrm{d}u$. For the mathematical derivations below it will be convenient to work with this limit, instead of the discrete version depending on n, although the interested readers are referred to similar derivations based on discrete monitoring in Carr, Lewis (2004). An advantage of the discrete derivation in that reference is that one can allow jumps in the underlying process, but we find the derivations more educational and elegant in the continuous case, pre-supposing that readers are well familiar with Itô calculus. Furthermore, in the discrete definition (1) one has to be careful with the indicator. Notice that in (1) we sum over all time periods $[t_{i-1}^n,t_i^{(n)}]$ for which the underlying lies within the corridor at the beginning, i.e. $X_{t_{i-1}}\in(a,b)$, whereas in practice one may further distinguish more accurately between such periods in which the underlying is completely within the corridor and such in which it enters or leaves the corridor. But all these subtleties disappear in the limit, making the exposition more concise and helping to focus on the essential qualitative understanding.

We distinguish between the two cases $X_t=S_t$ and $X_t=F_{t,T}$, because both cases are useful in practice and treated in the literature. On the one hand, Burgard, Torné (2019) focus on the case $X_t=S_t$ and note that this is what is "usually found in the market". On the other hand, Carr, Lewis (2004) consider the case $X_t=F_{t,T}$ for mathematical convenience and if one implements a corridor variance via the aforementioned replication strategy the case $X_t=F_{t,T}$ is also more convenient, hence practically useful. Even though the (limit) payoffs in both cases are different due to the fact that $1_{(a,b)}(F_{t,T})\neq 1_{(a,b)}(S_t)$ in general, the economic reasoning behind the practical use cases for corridor variance swaps is often very similar.

2 Derivation of replicating strategy

The following derivation is well-known, for instance see Carr, Lewis (2004); Burgard, Torné (2019) and the references therein. We begin with an important auxiliary lemma.

Lemma 2.1 (An auxiliary identity from calculus)
Let $h : [0,\infty) \to \mathbb{R}$ be twice differentiable (but $h^{''}$ needs not be continuous). Then for $x, t \geq 0$ we have

$$
\begin{aligned}
&h(x) - h(t) - h^{'}(t)\,(x-t) \\
&\quad = \int_0^t h^{''}(y)\,(y-x)_+\,\mathrm{d}y + \int_t^\infty h^{''}(y)\,(x-y)_+\,\mathrm{d}y.
\end{aligned}
$$

Proof
See the Appendix. ☐

Given parameters $0 < a < b < \infty$, we apply Lemma 2.1 to the function

$$
h_{a,b}(x) = \begin{cases} \frac{x}{a} + \log(a) - 1 & \text{, if } x < a \\ \log(x) & \text{, if } a \leq x \leq b \\ \log(b) + \frac{x}{b} - 1 & \text{, if } x > b \end{cases}.
$$

Intuitively, $h_{a,b}$ coincides with the logarithm on the interval $[a,b]$ and is extrapolated linearly outside, see Figure 1.

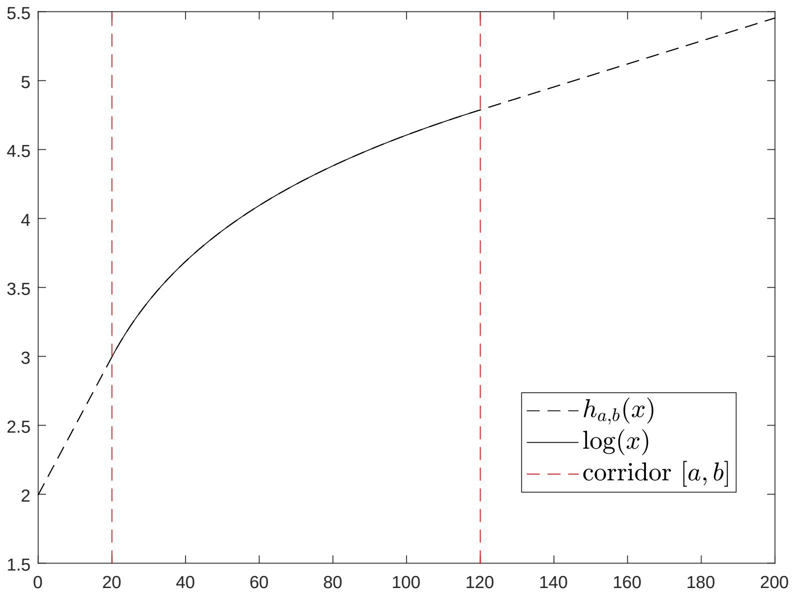

Figure 1: The function $h_{a,b}$ for $a = 50$ and $b = 120$.

We find from Lemma 2.1 that

$$
\begin{aligned}
&h_{a,b}(x) - h_{a,b}(t) - \frac{x-t}{\min\{b,\, \max\{a,t\}\}} \\
&\quad = -\int_0^t (y-x)_+ \frac{1_{(a,b)}(y)}{y^2}\,\mathrm{d}y - \int_t^\infty (x-y)_+ \frac{1_{(a,b)}(y)}{y^2}\,\mathrm{d}y.
\end{aligned}
\tag{2}
$$

With the help of this lemma we first discuss the replication of the realized variance leg, distinguishing the two cases $X_t = S_t$ (Subsection 2.1) and $X_t = F_{t,T}$ (Subsection 2.2). After that, in Subsection 2.3 we discuss the replication of the implied variance leg.

2.1 The corridor realized variance leg: based on the equity $X_t = S_t$

Itô’s formula implies that

$$\begin{aligned} h_{a,b}(S_T) - h_{a,b}(S_0) &= \int_0^T h^{'}_{a,b}(S_t)\,\mathrm{d}S_t + \frac{1}{2}\int_0^T h^{''}_{a,b}(S_t)\,\mathrm{d}[S,S]_t \\ &= \int_0^T \frac{1}{\min\{b,\,\max\{a,S_t\}\}}\,\mathrm{d}S_t - \frac{1}{2}\int_0^T \frac{1_{(a,b)}(S_t)}{S_t^2}\,\mathrm{d}[S,S]_t. \end{aligned}$$

If we apply (2) with $x = S_T$ and $t = S_0$, and we combine this with the last expression, we arrive at

$$\begin{aligned} \frac{1}{T}\int_0^T \frac{1_{(a,b)}(S_t)}{S_t^2}\,\mathrm{d}[S,S]_t &= \int_0^T \frac{2}{T\,\min\{b,\,\max\{a,S_t\}\}}\,\mathrm{d}S_t \\ &\quad + \frac{2}{T}\int_0^{S_0} \frac{1_{(a,b)}(K)\,(K-S_T)_+}{K^2}\,\mathrm{d}K \\ &\quad + \int_{S_0}^{\infty} \frac{1_{(a,b)}(K)\,(S_T-K)_+}{K^2}\,\mathrm{d}K \\ &= \underbrace{\int_0^T \frac{2\,e^{-\int_t^T r(s)-\delta\,\mathrm{d}s}}{T\,\min\{b,\,\max\{a,S_t\}\}}\,\mathrm{d}F_{t,T}}_{\text{(b)}} \\ &\quad + \underbrace{\int_0^T \frac{2\,(r(t)-\delta)\,S_t}{T\,\min\{b,\,\max\{a,S_t\}\}}\,\mathrm{d}t}_{\text{(c)}} \\ &\quad + \frac{2}{T}\Big\{\int_0^{S_0} \frac{1_{(a,b)}(K)\,(K-S_T)_+}{K^2}\,\mathrm{d}K \\ &\quad + \int_{S_0}^{\infty} \frac{1_{(a,b)}(K)\,(S_T-K)_+}{K^2}\,\mathrm{d}K\Big\} + \frac{2}{T}\,h^{'}_{a,b}(S_0)\,(S_0-S_T). \end{aligned}$$

This formula shows how to replicate the realized variance payoff, as will briefly be explained:

(a) The last two lines correspond to the payoff of a portfolio with standard puts and calls with maturity T (note that $S_0 - S_T = (S_0 - S_T)_+ - (S_T - S_0)_+$).

(b) The term with the $\mathrm{d}F_{t,T}$-integral corresponds to the PnL of a dynamic strategy in which at each t one holds a delta of $\frac{2}{T}\,e^{-\int_t^T r(s)-\delta\,\mathrm{d}s}$ units of the forward $F_{t,T}$.

(c) The line with the $\mathrm{d}t$-integral can be re-written as

$$\begin{aligned} &\int_0^T \frac{2\,(r(t)-\delta)\,S_t}{T\,\min\{b,\,\max\{a,S_t\}\}}\,\mathrm{d}t \\ &\quad = \int_0^T \frac{2\,(r(t)-\delta)}{T}\Big\{1 - \frac{1}{a}\,(a-S_t)_+ + \frac{1}{b}\,(S_t-b)_+\Big\}\,\mathrm{d}t, \end{aligned}$$

which equals the payout of a static portfolio that for each $t \in (0,T]$ consists of $\frac{2\,(r(t)-\delta)}{T}$ units of zero coupon bonds with maturity t, selling $\frac{2\,(r(t)-\delta)}{T\,a}$ puts with maturity t and strike a, and buying $\frac{2\,(r(t)-\delta)}{T\,b}$ calls with maturity t and strike b.

The reference Burgard, Torné (2019) derives a super-replicating hedging strategy for the realized variance that is simpler to implement, and in particular gets rid of the most inconvenient term (c) above.

2.2 The corridor realized variance leg: based on the forward $X_t = F_{t,T}$

An alternative possibility to get rid of the parts (b) and (c) in the preceding paragraph is explained in Carr, Lewis (2004), and we find it useful to recall this in the following. To this end, we consider the case $X_t = F_{t,T}$. By the analogous derivation as before, replacing S_t with $F_{t,T}$, we obtain (note that $F_{T,T} = S_T$)

$$
\begin{aligned}
\frac{1}{T}\int_0^T \frac{1_{(a,b)}(F_{t,T})}{F_{t,T}^2}\,\mathrm{d}[F_{.,T},F_{.,T}]_t &= \int_0^T \frac{2\,\mathrm{d}F_{t,T}}{T\,\min\{b,\,\max\{a,F_{t,T}\}\}}\\
&\quad+\frac{2}{T}\int_0^{F_{0,T}} \frac{1_{(a,b)}(K)\,(K-S_T)_+}{K^2}\,\mathrm{d}K\\
&\quad+\int_{F_{0,T}}^{\infty} \frac{1_{(a,b)}(K)\,(S_T-K)_+}{K^2}\,\mathrm{d}K\\
&= \frac{2}{T}\int_0^T h_{a,b}^{'}(F_{t,T})\,\mathrm{d}F_{t,T} + \frac{2}{T}\,h_{a,b}^{'}(F_{0,T})\,(F_{0,T}-S_T) \qquad (3)\\
&\quad+\frac{2}{T}\Big\{\int_0^{F_{0,T}} \frac{1_{(a,b)}(K)\,(K-S_T)_+}{K^2}\,\mathrm{d}K\\
&\quad+\int_{F_{0,T}}^{\infty} \frac{1_{(a,b)}(K)\,(S_T-K)_+}{K^2}\,\mathrm{d}K\Big\}
\end{aligned}
$$

Carr, Lewis (2004) now explain that the payoff in line (3) can be achieved by a continuous delta-hedging strategy for the static strip of puts and calls in the last two lines, when each delta is computed with a Black–Scholes model with volatility parameter $\sigma = 0$, and we state this as a lemma.

Lemma 2.2 (Re-interpretation of dynamic part)
Consider a portfolio of infinitely many vanilla puts and/or calls with maturity T, as follows: for each strike $K \in (a,b)$ we buy $2\,\mathrm{d}K/(T\,K^2)$ puts if $K < F_{0,T}$ and calls otherwise. We call this portfolio of puts and/or calls the static strip of puts/calls. We denote the Delta at time $t \in [0,T]$ of this portfolio by $D_t^{(\sigma,\infty)}$, assuming that the Delta is computed with the Black–Scholes model with parameter σ. It follows that

$$
\begin{aligned}
&-\int_0^T D_t^{(0,\infty)}\,e^{\int_t^T r(u)\,\mathrm{d}u}\,\mathrm{d}F_{t,T} \qquad (4)\\
&\qquad= \frac{2}{T}\int_0^T h_{a,b}^{'}(F_{t,T})\,\mathrm{d}F_{t,T} + \frac{2}{T}\,h_{a,b}^{'}(F_{0,T})\,(F_{0,T}-S_T).
\end{aligned}
$$

In words, the final outcome of the strategy of continuously delta-hedging the static strip of puts/calls with the Delta $D_t^{(0,\infty)}$ is equal to the expression (3).

Proof
See the Appendix. ☐

To better understand the appearance of the interest rate factor $\exp\big\{\int_{t_i}^T r(u)\,\mathrm{d}u\big\}$ in (4), it is helpful to look at the discretized version of the integral

$$
\int_0^T D_t^{(0,\infty)}\,e^{\int_t^T r(u)\,\mathrm{d}u}\,\mathrm{d}F_{t,T} \approx \sum_i \underbrace{D_{t_{i-1}}^{(0,\infty)}\,\Delta F_{t_{i-1},T}}_{\text{hedge gain in }[t_{i-1},t_i]}\; e^{\int_{t_i}^T r(u)\,\mathrm{d}u},
$$

and to notice that the hedge gain in period $[t_{i-1},t_i]$ is purely cash-settled (since implemented via forwards), hence can be put on the risk-free bank account where it accrues at the risk-free rate r from t_i until T.

Let us look a little closer at the practical implementation of the trade, and the discretization error.

Example 2.3 (Discretization error)
We consider the special case of a finite corridor with $b < F_{0,T} < \infty$ (the corridor is out-of-the money), and we assume we discretize the corridor via $a = K_0 < K_1 < \ldots < K_m = b$, where $\Delta K := (b-a)/m$ is an equi-distant spacing independent of $i = 1, \ldots, m$, and we denote $\bar{K}_i := (K_{i-1} + K_i)/2$ for each $i = 1, \ldots, m$. The put/call strip then only consists of out-of-the-money put options, to wit we assume for $i = 1, \ldots, m$ we buy $2\,\Delta K/(T\,\bar{K}_i^2)$ put options with strike $\bar{K}_i$. The difference between the payoffs at maturity T (which is assumed to be 0.28) of the continuous put strip and this discrete put strip is visualized in Figure 2. One realizes that already for moderately sized m the approximation is pretty good, hence the discretization error for the static strip of puts small.

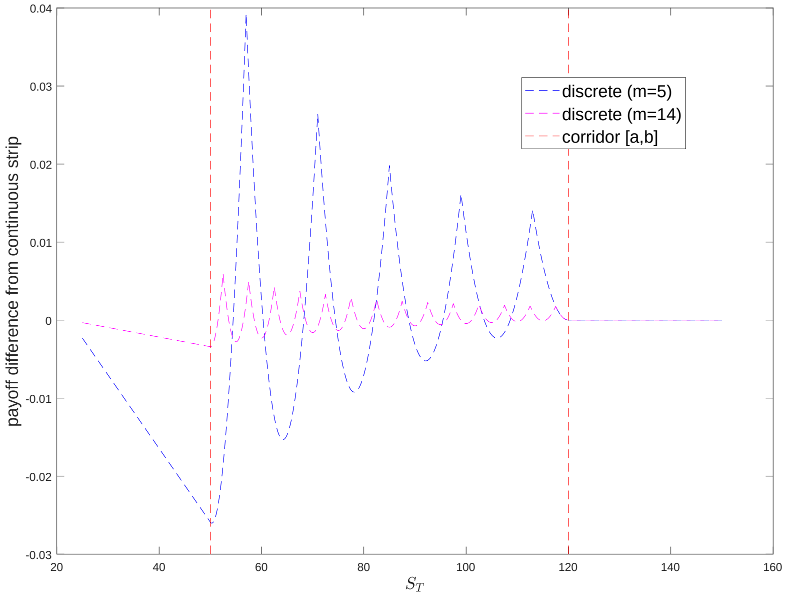

Figure 2: The difference between the payoffs at maturity of continuous and discretized put strips as a function of the final value S_T, with $a = 50$, $b = 120$ and two different m.

The zero-volatility Black–Scholes Delta of the put option with strike $\bar{K}_i$ at time t equals

$$\frac{\partial}{\partial F_{t,T}} P(F_{t,T}, \bar{K}_i, t) = -e^{-\int_t^T r(u)\,\mathrm{d}u}\, 1_{\{F_{t,T} < \bar{K}_i\}}.$$

Hence, the discrete strip has zero-volatility Delta equal to

$$D_t^{(0,m)} := -\frac{2\,\Delta K}{T}\, e^{-\int_t^T r(u)\,\mathrm{d}u} \sum_{i=1}^{m} \frac{1}{\bar{K}_i^2}\, 1_{\{F_{t,T} < \bar{K}_i\}}, \qquad (5)$$

which is a step function in the underlying $F_{t,T}$, see the top plot in Figure 3. Unfortunately, in the zero-volatility case we cannot compute the Gamma of the strip, even though theory suggests that one has constant Gamma within the corridor. But we can view (5) as the limit $\sigma \searrow 0$ of the same Delta computed in a Black–Scholes model with volatility $\sigma > 0$ (instead of zero). To wit, this value is given by

$$\begin{aligned} D_t^{(\sigma,m)} &:= -\frac{2\,\Delta K\, e^{-\int_t^T r(u)\,\mathrm{d}u}}{T} \times \\ &\times \sum_{i=1}^{m} \frac{1}{\bar{K}_i^2}\,\Phi\Big(\frac{\log(\bar{K}_i) - \log(F_{t,T}) - \frac{\sigma^2}{2}\,(T-t)}{\sigma\,\sqrt{T-t}}\Big), \end{aligned} \tag{6}$$

and has a derivative with respect to $F_{t,T}$ (the Gamma) given by

$$\begin{aligned} G_t^{(\sigma,m)} &:= \frac{\sqrt{2}\,\Delta K\, e^{-\int_t^T r(u)\,\mathrm{d}u}}{T\,F_{t,T}\,\sqrt{\pi\,(T-t)}\,\sigma} \times \\ &\times \sum_{i=1}^{m} \frac{1}{\bar{K}_i^2}\,\exp\Big\{-\frac{\big(\log(F_{t,T}/\bar{K}_i) + \frac{\sigma^2}{2}\,(T-t)\big)^2}{2\,\sigma^2\,(T-t)}\Big\}. \end{aligned} \tag{7}$$

Figure 3 illustrates the values $D_t^{(\sigma,m)}$ and $G_t^{(\sigma,m)}\,F_{t,T}^2$ as functions of the underlying $F_{t,T}$ for different values of σ and m. It is now an interesting observation that the desired theoretical Delta $D_t^{(0,\infty)}$, that provides constant Gamma exposure within the corridor, is approximated better by $D_t^{(\sigma,m)}$ for some $\sigma > 0$ than by $D_t^{(0,m)}$. This means that the error that is introduced to the desired Delta when changing from the continuous case $m = \infty$ to a finite $m < \infty$ can be alleviated by simultaneously changing from the zero-volatility case $\sigma = 0$ to some positive $\sigma > 0$. Figure 3 visualizes this effect.

2.3 The corridor implied variance leg

Since $\mathrm{d}[X,X]_t = \sigma_t^2\,X_t^2\,\mathrm{d}t$, the expected value of the realized corridor variance under $\mathbb{Q}$ equals

$$\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T}\int_0^T \frac{1_{(a,b)}(X_t)}{X_t^2}\,\mathrm{d}[X,X]_t\Big] = \frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}\Big[1_{(a,b)}(X_t)\,\sigma_t^2\Big]\,\mathrm{d}t.$$

The fair implied corridor variance, which justifies a zero value of the corridor variance swap at inception, is given by

$$\bar{\sigma}_{CVS}^2 = \frac{\int_0^T \mathbb{E}^{\mathbb{Q}}\Big[1_{(a,b)}(X_t)\,\sigma_t^2\Big]\,\mathrm{d}t}{\int_0^T \mathbb{E}^{\mathbb{Q}}\Big[1_{(a,b)}(X_t)\Big]\,\mathrm{d}t}.$$

We have seen in the preceding subsections how the realized corridor variance can be replicated by a portfolio of puts and calls, an equity forward and some delta-trading. Now we investigate how the payoff $\frac{1}{T}\int_0^T 1_{(a,b)}(X_t)\,\mathrm{d}t$ in the implied corridor variance leg can be replicated. To this end, we observe that

$$1_{(a,b)}(X_t) = 1_{\{X_t < b\}} - 1_{\{X_t \leq a\}}. \tag{8}$$

Fixing some $N \in \mathbb{N}$, if we buy $2\,N$ puts with strike $K - \frac{1}{N}$ and sell N puts with strike $K + \frac{1}{N}$, both with maturity t, the payoff at

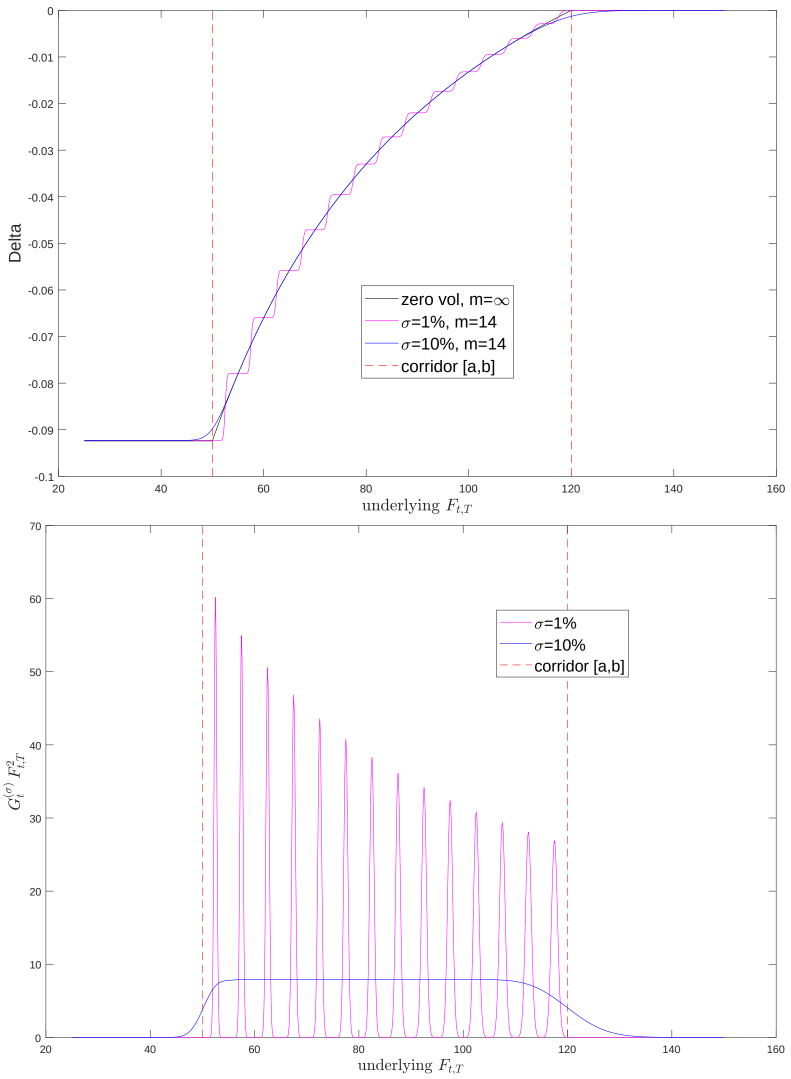

Figure 3: Top: The Deltas D_t and $D_t^{(\sigma)}$ as functions of $F_{t,T}$ for $a = 50$ and $b = 120$ and $\Delta K = 5$ (i.e. $m = 14$) for different σ. Top: The Gammas $G_t^{(\sigma)}\,F_{t,T}^2$ as functions of $F_{t,T}$ for $a = 50$ and $b = 120$ and $\Delta K = 5$ (i.e. $m = 14$) for different σ.

t of this portfolio equals

$$2\,N\left\{\left(K-\frac{1}{N}-X_t\right)_+-\left(K+\frac{1}{N}-X_t\right)_+\right\}$$
$$=\begin{cases}-1 & \text{, if } X_t < K-\frac{1}{N}\\ 2\,N\,(X_t-K)-\frac{1}{2} & \text{, if } K-\frac{1}{N}\leq X_t\leq K+\frac{1}{N}\\ 0 & \text{, if } X_t > K+\frac{1}{N}\end{cases}$$
$$\xrightarrow{N\to\infty}\ -1_{\{X_t\leq K\}}.$$

Consequently, we can approximate the digital payoff (8) via standard put options with the following portfolio:

- Buy $2\,N$ puts with strike $a-\frac{1}{N}$ and $2\,N$ puts with strike $b+\frac{1}{N}$, both with maturity t.
- Sell $2\,N$ puts with strike $a+\frac{1}{N}$ and $2\,N$ puts with strike $b-\frac{1}{N}$, both with maturity t.

With a partition $0 = t_0 < t_1 < \ldots < t_n = T$ this implies that

$$\begin{aligned}\frac{1}{T}\int_0^T 1_{(a,b)}(X_t)\,\mathrm{d}t &\approx \sum_{i=1}^n 1_{(a,b)}(X_{t_{i-1}})\,\frac{t_i - t_{i-1}}{T}\\ &= \frac{2\,N}{T}\sum_{i=1}^n (t_i - t_{i-1})\Big\{\Big(a - \frac{1}{N} - X_{t_{i-1}}\Big)_+ - \Big(a + \frac{1}{N} - X_{t_{i-1}}\Big)_+\\ &\qquad + \Big(b + \frac{1}{N} - X_{t_{i-1}}\Big)_+ - \Big(b - \frac{1}{N} - X_{t_{i-1}}\Big)_+\Big\}.\end{aligned}$$

If $a = -\infty$, the puts with strike $a \pm \frac{1}{N}$ are not required.

3 The Delta

First of all, obvious but still noteworthy, in a Black-Scholes market the (corridor) variance swap is a zero contract, since realized and implied legs are identical almost surely. Consequently, in a Black–Scholes market the Delta is exactly zero. In particular, this implies that (the Delta of) implied and realized legs are identical in a Black–Scholes market.

We investigate the Delta of a corridor variance swap under the assumption that the underlying forward $F_{t,T}$ follows a so-called JDCEV model in the spirit of Carr, Linetsky (2006). The latter is a local volatility model extended by credit risk, but for the sake of the present demonstration the particular model choice is irrelevant. The only important thing is that it deviates from Black–Scholes and hence the corridor variance swap has a Delta. For the sake of an educational exposition, we reconsider the out-of-the-money corridor variance swap from Example 2.3. The realized leg is discretized with parameters $m = 14$ and $\sigma = 10\%$, which implies a very small error according to Figure 3. The implied leg is realized according to $t_i - t_{i-1} = T/8$. The implied volatility surface for the applied JDCEV model is depicted in Figure 4, so that one can clearly observe the strong deviation from the Black–Scholes case.

Figure 5 depicts both the Delta of the realized leg and the Delta of the implied leg. There are two things to observe. First, the Deltas of realized and implied corridor variance are unequal, hence the corridor variance swap has a non-zero Delta. In particular, the corridor variance swap has a significantly positive Delta around the upper end of the corridor. Second, the qualitative shape of the Delta is identical for the realized and the implied corridor variance legs (and, again, they would coincide in a Black–Scholes model). The Delta is close to zero except it is positive around the lower end point of the corridor and negative around the upper end point of the corridor. This means that each single leg of a corridor variance swap intuitively profits (suffers) from an entry into (exit out of) the corridor. In total, an out-of-the-money corridor variance swap exhibits a positive Delta around the upper corridor end point and a negative Delta around the lower corridor end point. The title of the bottom plot depicts the implied strike volatility $\bar{\sigma}_{CVS} = 30.73\%$, which is also computed in the applied JDCEV model.

Appendix

We collect technical proofs for the statements made in the main body of this article.

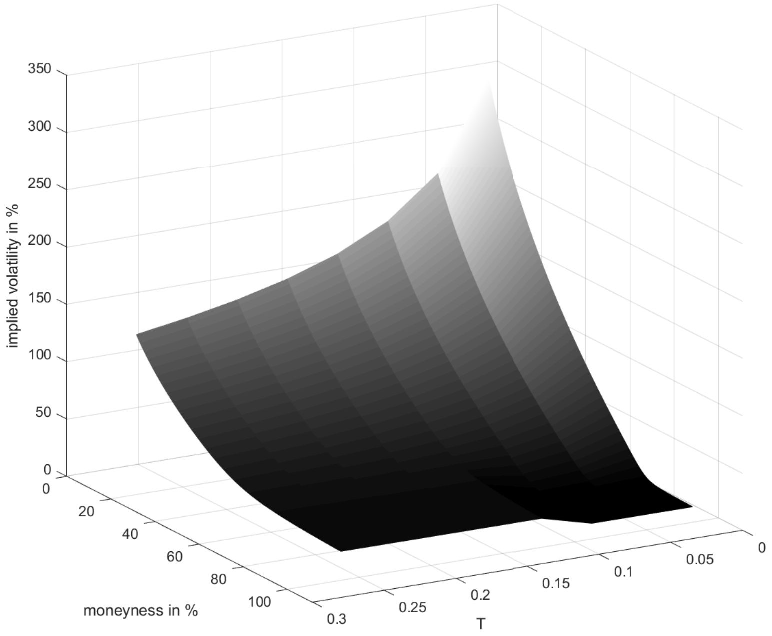

Figure 4: Implied volatility surface of JDCEV model that is applied to compute the Delta of the corridor variance swap here.

Proof (of Lemma 2.1)
We compute the integral $\int_x^t h^{''}(y)\,(y-x)\,\mathrm{d}y$ in two different ways. First, we observe $y-x=(y-x)_+-(x-y)_+$ and obtain

$$\begin{aligned}
&\int_x^t h^{''}(y)\,(y-x)\,\mathrm{d}y\\
&\quad=\int_x^t h^{''}(y)\,(y-x)_+\,\mathrm{d}y-\int_x^t h^{''}(y)\,(x-y)_+\,\mathrm{d}y\\
&\quad=\int_x^t h^{''}(y)\,(y-x)_+\,\mathrm{d}y+\int_t^x h^{''}(y)\,(x-y)_+\,\mathrm{d}y\\
&\quad=\int_0^t h^{''}(y)\,(y-x)_+\,\mathrm{d}y+\int_t^\infty h^{''}(y)\,(x-y)_+\,\mathrm{d}y.
\end{aligned}$$

Second, we apply integration by parts to obtain

$$\begin{aligned}
\int_x^t h^{''}(y)\,(y-x)\,\mathrm{d}y &= h^{'}(t)\,(t-x)-\int_x^t h^{'}(y)\,\mathrm{d}y\\
&= h^{'}(t)\,(t-x)-h(t)+h(x).
\end{aligned}$$

A combination of the two expressions implies the claim. □

Proof (of Lemma 2.2)
Under the assumption $\sigma=0$, the price and delta of a European put option with strike K and maturity T are equal to

$$\begin{aligned}
P(F_{t,T},K,t) &= e^{-\int_t^T r(u)\,\mathrm{d}u}\,(K-F_{t,T})_+,\\
\frac{\partial}{\partial F_{t,T}}P(F_{t,T},K,t) &= -e^{-\int_t^T r(u)\,\mathrm{d}u}\,1_{\{F_{t,T}<K\}}.
\end{aligned}$$

Note that the put option's delta is undefined at K, but this is irrelevant for the derivation due to the continuous integration. The

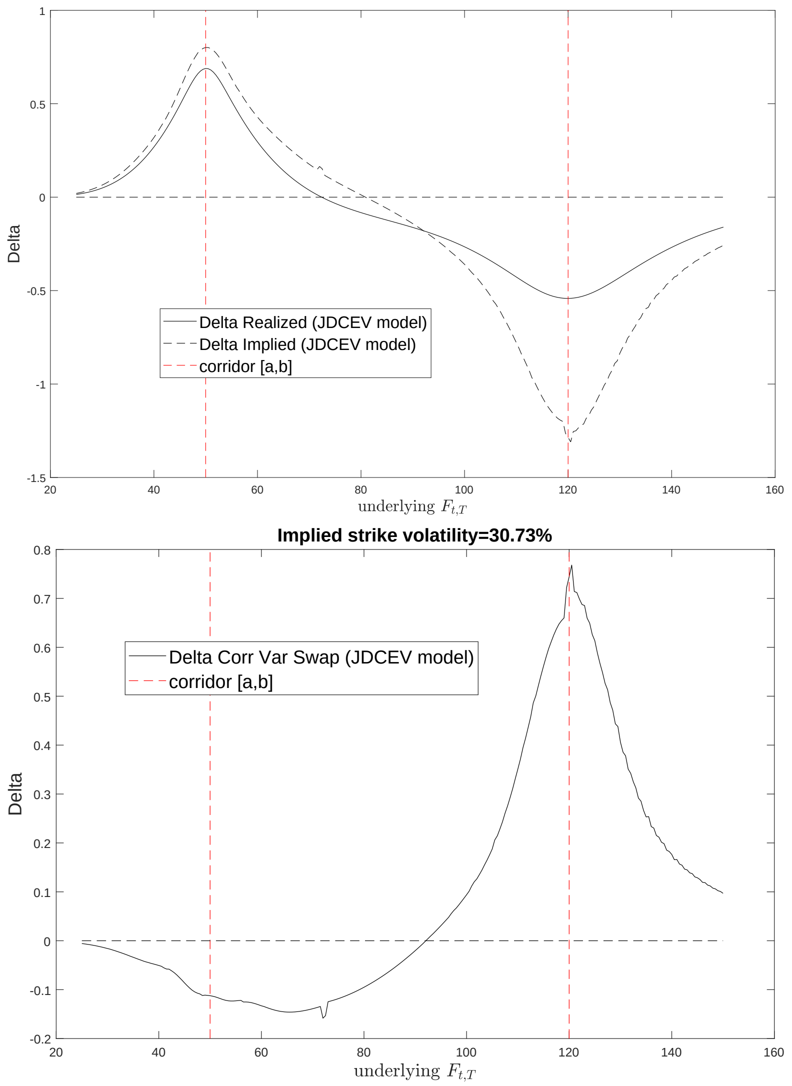

Figure 5: Top: The Delta of realized and implied corridor variance legs, where $[a, b] = [50, 125]$. Bottom: The Delta of the corridor variance swap with $[a, b] = [50, 125]$. The blue dotted line depicts the current forward value. The model for computing the Deltas is a linear local volatility model.

respective formulas for the call options are analogously given by

$$C(F_{t,T}, K, t) = e^{-\int_t^T r(u)\,\mathrm{d}u}\,(F_{t,T} - K)_+,$$
$$\frac{\partial}{\partial F_{t,T}} C(F_{t,T}, K, t) = e^{-\int_t^T r(u)\,\mathrm{d}u}\,1_{\{F_{t,T}>K\}}.$$

With these formulas, we are able to compute the delta for the put/call strip at time t, to wit

$$\begin{aligned}
&\frac{\partial}{\partial F_{t,T}}\Big\{\frac{2}{T}\int_0^{F_{0,T}}\frac{1_{(a,b)}(K)\,P(F_{t,T},K,t)}{K^2}\,\mathrm{d}K \\
&\qquad+\frac{2}{T}\int_{F_{0,T}}^{\infty}\frac{1_{(a,b)}(K)\,C(F_{t,T},K,t)}{K^2}\,\mathrm{d}K\Big\} \\
&=-\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\int_0^{F_{0,T}}\frac{1_{(a,b)\cap(F_{t,T},\infty)}(K)}{K^2}\,\mathrm{d}K \\
&\qquad+\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\int_{F_{0,T}}^{\infty}\frac{1_{(a,b)\cap(0,F_{t,T})}(K)}{K^2}\,\mathrm{d}K=(*).
\end{aligned}$$

From here, there are two possibilities, either $F_{t,T}\leq F_{0,T}$ or $F_{t,T}>F_{0,T}$. Assuming the first case $F_{t,T}\leq F_{0,T}$ holds true, we obtain

$$\begin{aligned}
(*)&=-\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\int_{F_{t,T}}^{F_{0,T}}\frac{1_{(a,b)}(K)}{K^2}\,\mathrm{d}K \\
&=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\Big(\frac{1}{\max\{\min\{b,F_{0,T}\},a\}}-\frac{1}{\max\big\{\min\{b,F_{t,T}\},a\big\}}\Big) \\
&=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\big(h'_{a,b}(F_{0,T})-h'_{a,b}(F_{t,T})\big).
\end{aligned}$$

And assuming the second case $F_{t,T}>F_{0,T}$ holds true, we obtain analogously

$$\begin{aligned}
(*)&=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\int_{F_{0,T}}^{F_{t,T}}\frac{1_{(a,b)}(K)}{K^2}\,\mathrm{d}K \\
&=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\Big(\frac{1}{\max\{\min\{F_{0,T},b\},a\}}-\frac{1}{\max\{\min\big\{F_{t,T},b\big\},a\}}\Big) \\
&=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\big(h'_{a,b}(F_{0,T})-h'_{a,b}(F_{t,T})\big).
\end{aligned}$$

Summing up, the delta of the strip (assuming zero volatility) equals

$$D_t^{(0,\infty)}:=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\big(h'_{a,b}(F_{0,T})-h'_{a,b}(F_{t,T})\big). \tag{9}$$

Notice that the associated Gamma, i.e. the derivative of the Delta $D_t^{(0,\infty)}$ with respect to the underlying, equals

$$G_t^{(0,\infty)}:=-\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\,h''_{a,b}(F_{t,T})=\frac{2}{T}\,e^{-\int_t^T r(u)\,\mathrm{d}u}\,\frac{1_{(a,b)}(F_{t,T})}{F_{t,T}^2},$$

meaning that the choice of Delta $D_t^{(0,\infty)}$ gives constant Gamma exposure $G_t^{(0,\infty)}\,F_{t,T}^2$ within the corridor (and zero Gamma exposure outside). Now the expression $-\int_0^T D_t^{(0,\infty)}\,e^{\int_t^T r(u)\,\mathrm{d}u}\,\mathrm{d}F_{t,T}$ is precisely the final payoff of the aforementioned dynamic delta-hedge of the static strip of puts and calls under zero volatility assumption, and it is now easy to verify from (9) that the expression $-\int_0^T D_t^{(0,\infty)}\,e^{\int_t^T r(u)\,\mathrm{d}u}\,\mathrm{d}F_{t,T}$ is equal to (3), as claimed. ☐

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