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An introduction to the VIX and the pricing of VIX calls

Jan-Frederik Mai
XAIA Investment GmbH
Sonnenstraße 19, 80331 München, Germany
jan-frederik.mai@xaia.com

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Abstract This article provides the mathematical background to understand the definition of the VIX index. The latter is designed to track the market's expectation of the realized variance of the S&P 500 stock price index over the upcoming month. Our derivations are largely motivated by standard references that explain the replication of variance swaps, such as Demeterfi et al. (1999); Carr, Madan (2001); Carr, Lewis (2004); Duembgen et al. (2015). The readers are assumed to have basic knowledge about stochastic analysis and the Black–Scholes stock pricing model, which is the content of a standard university lecture in financial mathematics. Finally, the article also recalls from Arai (2019) how to implement pricing methods for VIX calls within the Barndorff-Nielsen Shephard stochastic volatility model. This methodology requires the reader to be familiar with Lévy processes and Fourier option pricing, which is only very briefly recalled in order to keep the article compact.

1 Delta-hedged European claim in the Black–Scholes model

We recall the Black–Scholes model with stock price process

$$S_t = S_0\, e^{\left(\mu-\frac{\sigma^2}{2}\right)\, t+\sigma\, W_t}, \quad t \geq 0,$$

where W denotes a standard Brownian motion, $\mu \in \mathbb{R}$ a drift rate and $\sigma > 0$ a volatility parameter. We further asssume there is a risk-free bank account $B_t = \exp\{r\, t\}$, $t \geq 0$, with risk-free rate $r \in \mathbb{R}$. We denote by $(\mathcal{F}_t)$ the natural filtration of S, and by $X = h(S_T)$ a European claim on the stock with maturity T and payoff function h. Note that X is $\mathcal{F}_T$-measurable. Arbitrage pricing theory tells us that the unique arbitrage-free price of the claim X at time $t \in [0,T]$ equals

$$V_t^X = e^{-r\,(T-t)}\, \mathbb{E}_{\mathbb{Q}}[h(S_T)\,|\,\mathcal{F}_t],$$

where $\mathbb{Q}$ denotes the unique risk-neutral pricing measure, under which the stock price process has drift r instead of μ. The Black–Scholes PDE states that there exists a smooth function $V = V(t,s)$ satisfying

$$V_t + \frac{1}{2}\,\sigma^2\, s^2\, V_{ss} + r\, s\, V_s - r\, V = 0, \quad V(T,s) = h(s), \tag{1}$$

such that $V_t^X = V(t, S_t)$. Using this identity below in $(*)$, we observe with Itô's formula that

$$\begin{aligned}\mathrm{d}V(t,S_t) &= \Big(\mu\, S_t\, V_s(t,S_t) + V_t(t,S_t) + \frac{1}{2}\,\sigma^2\, S_t^2\, V_{ss}(t,S_t)\Big)\,\mathrm{d}t \\ &\quad + \sigma\, S_t\, V_s(t,S_t)\,\mathrm{d}W_t \\ &= V_s(t,S_t)\,\mathrm{d}S_t + \Big(V_t(t,S_t) + \frac{1}{2}\,\sigma^2\, S_t^2\, V_{ss}(t,S_t)\Big)\,\mathrm{d}t \\ &\overset{(*)}{=} V_s(t,S_t)\,\mathrm{d}S_t + \frac{V(t,S_t) - V_s(t,S_t)\, S_t}{B_t}\,\mathrm{d}B_t.\end{aligned}$$

This shows that the trading strategy $\varphi_t = (\varphi_t^{(S)}, \varphi_t^{(B)})$ is self-financing, where

$$\varphi_t^{(S)} = V_s(t, S_t), \quad \varphi_t^{(B)} = \frac{V(t, S_t) - V_s(t, S_t)\, S_t}{B_t}.$$

Note that V_s is the delta of the claim, so that the dynamics of the (continuously) delta-hedged option position amounts to

$$\begin{aligned} \mathrm{d}V(t, S_t) - V_s(t, S_t)\,\mathrm{d}S_t &= \varphi_t^{(B)}\,\mathrm{d}B_t \\ &= \underbrace{V_t(t, S_t)}_{\text{theta}}\,\mathrm{d}t + \underbrace{\frac{1}{2}\sigma^2\, S_t^2\, V_{ss}(t, S_t)}_{\text{gamma}}\,\mathrm{d}t. \end{aligned}$$

The last equality depicts a decomposition into theta (negative for convex claims) and gamma (positive for convex claims). Intuitively, as a buyer of the delta-hedged position for a convex claim (like standard put and call) the theta runs against you (time value loss) while the gamma runs in your favor. Figure 1 depicts the mechanics of the realized PnL for a put option, where we write $V = P$, i.e. $h(s) = (K - s)_+$ for some strike price $K > 0$.

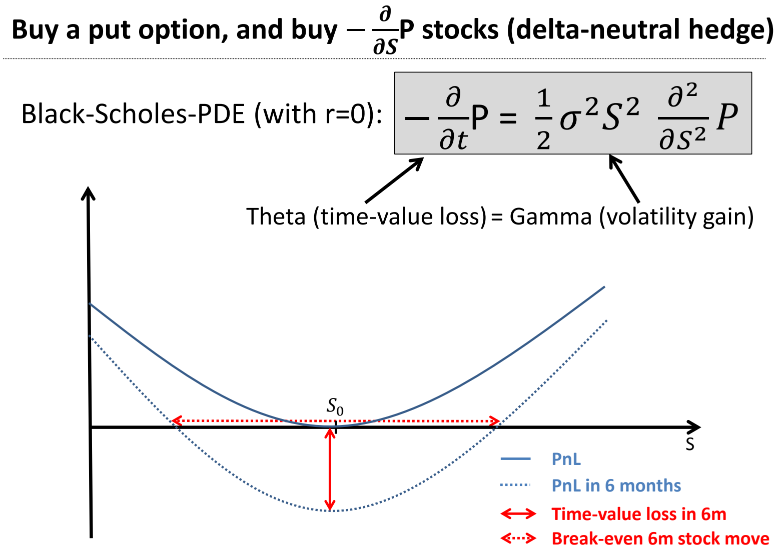

Figure 1: Illustration of the "theta vs. gamma"-play when realizing the PnL of a delta-hedged put position in the Black–Scholes model.

2 Delta-hedged European claim for arbitrary Itô process

We explain how the intuition from the Black–Scholes model carries over also to more general (model-free) considerations. In a more general diffusion model framework beyond Black–Scholes, assuming the stock price process to be an Itô process, the same derivation holds and the aforementioned delta-theta identity reads

$$\mathrm{d}V(t, S_t) - V_s(t, S_t)\,\mathrm{d}S_t = \underbrace{V_t(t, S_t)}_{\text{theta}}\,\mathrm{d}t + \underbrace{\frac{1}{2}\, V_{ss}(t, S_t)\,\mathrm{d}[S, S]_t}_{\text{gamma}},$$

where $[S,S]$ equals the quadratic variation of S. We furthermore know that

$$\lim_{n\to\infty}\sum_{i=1}^{n} V_{ss}(t_{i-1}^{(n)}, S_{t_{i-1}^{(n)}})\,(S_{t_i^{(n)}} - S_{t_{i-1}^{(n)}})^2 = \int_0^T V_{ss}(t, S_t)\,\mathrm{d}[S,S]_t,$$

where the mesh of the discrete time grids $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T$ tends to zero with n, and the limit holds in probability. Performing the delta-hedging strategy on a discrete time grid, we thus approximately obtain the realized PnL

$$\begin{aligned}&\sum_{i=1}^{n} V(t_i, S_{t_i}) - V(t_{i-1}, S_{t_{i-1}}) - V_s(t_{i-1}, S_{t_{i-1}})\,(S_{t_i} - S_{t_{i-1}})\\ &\approx \sum_{i=1}^{n} \frac{1}{2} V_{ss}(t_{i-1}, S_{t_{i-1}})\,(S_{t_i} - S_{t_{i-1}})^2 + V_t(t_{i-1}, S_{t_{i-1}})\,(t_i - t_{i-1}).\end{aligned} \tag{2}$$

Now we consider the theta expression in the last equation. The option is bought at trade inception $t = 0$ for a price that is quoted in terms of the implied volatility parameter σ in the Black–Scholes model (depending on T), even though the Black–Scholes model is not appropriate in reality. However, we might not make a big error when approximating the (real) theta $V_t(t_{i-1}, S_{t_{i-1}})$ in the last equation by the respective expression obtained from solving the Black–Scholes PDE (1) for the theta. Concretely, we assume (1) holds approximately also for the general model value function V, with the parameter σ that is determined from the Black–Scholes model by the observed option price at $t = 0$. With this approximation we obtain

$$\begin{aligned}V_t(t_{i-1}, S_{t_{i-1}}) \approx& -\frac{1}{2} V_{ss}(t_{i-1}, S_{t_{i-1}})\,\sigma^2\, S_{t_{i-1}}^2\\ &- r\left(S_{t_{i-1}}\, V_s(t_{i-1}, S_{t_{i-1}}) - V(t_{i-1}, S_{t_{i-1}})\right).\end{aligned}$$

And consequently, the realized PnL of the delta-hedged position is approximately equal to

$$\begin{aligned}&\sum_{i=1}^{n} V(t_i, S_{t_i}) - V(t_{i-1}, S_{t_{i-1}}) + V_s(t_{i-1}, S_{t_{i-1}})\,(S_{t_i} - S_{t_{i-1}})\\ &\approx \underbrace{-\sum_{i=1}^{n} r\,(t_i - t_{i-1})\,\{S_{t_{i-1}}\, V_s(t_{i-1}, S_{t_{i-1}}) - V(t_{i-1}, S_{t_{i-1}})\}}_{(*)}\\ &+ \sum_{i=1}^{n} \underbrace{\frac{1}{2} S_{t_{i-1}}^2\, V_{ss}(t_{i-1}, S_{t_{i-1}})}_{\text{gamma weight}} \Bigg(\underbrace{\frac{(S_{t_i} - S_{t_{i-1}})^2}{S_{t_{i-1}}^2}}_{\text{realized var.}} - \underbrace{\sigma^2\,(t_i - t_{i-1})}_{\text{implied var.}} \Bigg).\end{aligned}$$

The term $(*)$ corresponds to an opportunity loss, because the amount in $\{.\}$-brackets is bound in the trade over the period $[t_{i-1}, t_i]$ and hence does not earn the rate r, which it would if this amount would be held in cash over the same period. The implied variance term is essentially the market's expectation on the realized variance over the interval $[0, T]$ at trade inception $t = 0$. For positive gamma (like standard put and call) this formula shows that the trade PnL tends to be positive when the realized variance is larger than what the market expects (implicitly via its option price).

3 Intuition and definition of the VIX

One of the crucial insights from the derivation in the preceding section is the following: if one can manage to define a payoff function h such that the gamma weight $s^2\,V_{ss}$ is a constant, the realized PnL of the delta-hedged position equals a linear function in the realized variance. In particular, under the assumption of no arbitrage the expected PnL of the delta-hedged position should be equal to zero. If $s^2\,V_{ss}$ was a constant, this implies that the implied variance parameter σ^2 of the respective claim essentially gives the market's expectation of the realized variance over $[0,T]$. The essential idea of the VIX index is to provide a quantity that equals precisely this parameter σ. Consequently, we have to look for a payoff function h that makes the gamma weight constant.

To this end, we consider the logarithmic payoff $h(s) = \log(s) - \log(S_0)$. The intuition is that the second derivative of h is proportional to $1/s^2$, which carries over to the second derivative of V. Thus, the associated value function leads to a constant gamma weight in the realized PnL of the delta-hedged position for the associated claim. In the Appendix in Section 5.1, we verify that this is indeed the case, and we will therefore work with the logarithm in the following derivation. On the one hand, we observe with Itô's formula that

$$h(S_T) = \log(S_T/S_0) = \int_0^T \frac{1}{S_t}\,\mathrm{d}S_t - \underbrace{\frac{1}{2}\int_0^T \frac{1}{S_t^2}\,\mathrm{d}[S,S]_t}_{\text{realized variance}}\,.$$

On the other hand, for arbitrary $K_* > 0$ one may check that

$$\log\Big(\frac{S_T}{K_*}\Big) = \frac{S_T - K_*}{K_*} - \int_0^{K_*} \frac{(K - S_T)_+}{K^2}\,\mathrm{d}K - \int_{K_*}^{\infty} \frac{(S_T - K)_+}{K^2}\,\mathrm{d}K.$$

Combining the two equations, we may represent the realized variance as

$$\frac{1}{2}\int_0^T \frac{1}{S_t^2}\,\mathrm{d}[S,S]_t = \int_0^T \Big(\frac{1}{S_t} - \frac{1}{K_*}\Big)\,\mathrm{d}S_t + \frac{K_* - S_0}{K_*} + \int_0^{K_*} \frac{(K - S_T)_+}{K^2}\,\mathrm{d}K + \int_{K_*}^{\infty} \frac{(S_T - K)_+}{K^2}\,\mathrm{d}K + \log\Big(\frac{S_0}{K_*}\Big).$$

We denote by $P(K)$ and $C(K)$ the market prices for a European put and call option with maturity T and strike K on the underlying S. Multiplying both sides of the last equation with $2/T$ and taking the expectation with respect to $\mathbb{Q}$, we obtain[1]

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{T}\int_0^T \frac{1}{S_t^2}\,\mathrm{d}[S,S]_t\Big] \approx -\frac{1}{T}\Big(\frac{S_0\,e^{r\,T}}{K_*} - 1\Big)^2 + \frac{2}{T}\Big\{\int_0^{K_*} \frac{e^{r\,T}\,P(K)}{K^2}\,\mathrm{d}K + \int_{K_*}^{\infty} \frac{e^{r\,T}\,C(K)}{K^2}\,\mathrm{d}K\Big\}.$$

[1] Here, we once use the Taylor approximation $\log(x) \approx (x-1) - \frac{1}{2}\,(x-1)^2$, applied with $x = S_0\,e^{r\,T}/K_*$.

Since we have the limit in probability

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(S_{t_i^{(n)}}-S_{t_{i-1}^{(n)}})^2}{S^2_{t_{i-1}^{(n)}}}=\int_0^T\frac{1}{S_t^2}\,\mathrm{d}[S,S]_t,$$

the left-hand side of the last equation equals the market's expectation on the realized instantaneous variance (which would be equal to the constant σ^2 in the Black–Scholes model), while the right-hand side is a value that can be read off from market-observed European put and call prices. The idea of the VIX index is to be an observable quantity that measures (the square root of) this expectation. With the previous derivation in mind, the VIX index is defined as a discrete approximation of the integrals in the last formula, to wit

$$\text{VIX}=100\times\sqrt{\frac{2}{T}\sum_i\frac{K_{i+1}-K_{i-1}}{2}\,\frac{e^{r\,T}\,O(K_i)}{K_i^2}-\frac{1}{T}\Big(\frac{F}{K_0}-1\Big)^2},$$

where $F=S_0\,\exp\{r\,T\}$ denotes the forward price with maturity T, the strike K_0 equals the first observable strike level below the forward price F, and $O(K_i)=P(K_i)$ if $K_i<F$ (resp. $O(K_i)=C(K_i)$ if $K_i\geq F$). The VIX index refers to the S&P 500 stock price index, and the maturity T is approximately 30 days.

4 Pricing VIX calls

We have learned in the preceding sections in which sense the VIX index provides a justified approximation to the (purely mathematical) quantity σ_t of a stochastic volatility model, the precise proxy being given below in (3). Judging from observations of the time series of the VIX index, Barndorff–Nielsen, Shephard (2001) proposed to model the stochastic volatility process σ_t and the stock price process S as[2]

$$S_t=S_0\,\exp\Big\{\big(r+\Psi_{\nu_Z}(-\rho)\big)\,t-\frac{1}{2}\int_0^t\sigma_u^2\,\mathrm{d}u+\int_0^t\sigma_u\,\mathrm{d}W_s+\rho\,Z_{\lambda\,t}\Big\},$$

$$\sigma_t^2=e^{-\lambda\,t}\,\sigma_0^2+e^{-\lambda\,t}\int_0^t e^{\lambda\,s}\,\mathrm{d}Z_{\lambda\,s},\quad t\geq 0,$$

where Z denotes a driftless Lévy subordinator and $\lambda>0,\rho\leq 0$ are model parameters. We denote by ν_Z the Lévy measure of $\{Z_{\lambda\,t}\}_{t\geq 0}$ and by

$$\Psi_{\nu_Z}(x)=\int_0^\infty\Big(1-e^{-x\,y}\Big)\,\nu(\mathrm{d}y)$$

its associated Bernstein function. The parameter ρ introduces a leverage effect, because non-negative ρ implies that the stock price has a downward jump at the same time when the volatility spikes. The parameter λ controls how quickly the volatility declines after a spike. Barndorff–Nielsen, Shephard (2001) explain that the unusual timing $Z_{\lambda\,t}$ (instead of Z_t) is chosen deliberately so that the probability law of σ_t^2 is invariant with respect to the parameter λ. If S is a model for the S&P 500 index, this implies

[2] Note that the drift is such that $e^{-r\,t}\,S_t$ is a martingale.

(according to the preceding paragraph) that the VIX index is approximately equal to the mathematical quantity

$$\text{VIX}_t \approx \sqrt{\frac{1}{T}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^{t+T} \sigma_u^2 \,\mathrm{d}u \,\Big|\, \mathcal{F}_t\Big]} = \sqrt{b_V\,\sigma_t^2 + c_V}, \qquad (3)$$

where the last equality is derived in Arai (2019) and the constants b_V and c_V are given by

$$b_V = \frac{1-e^{-\lambda T}}{\lambda T},$$
$$c_V = \Big(\frac{1-b_V}{\lambda} - 2\,\rho\Big)\int_0^\infty x\,\nu_Z(\mathrm{d}x) - 2\,\Psi_{\nu_Z}(-\rho).$$

We note that $\int_0^\infty x\,\nu_Z(\mathrm{d}x) = \mathbb{E}_{\mathbb{Q}}[Z_\lambda]$ equals the expected value of Z_λ. In intuitive terms, this expectation increases in the frequency of jumps as well as in the expected size of jumps, which are both quantities that are typically modeled via parameters. Consequently, the current observation of VIX_0 implies a restriction on the model parameters according to

$$\sqrt{\Big(\frac{1-b_V}{\lambda} - 2\,\rho\Big)\,\mathbb{E}_{\mathbb{Q}}[Z_\lambda] - 2\,\Psi_{\nu_Z}(-\rho)} < \text{VIX}_0. \qquad (4)$$

In particular, this is an upper bound for the leverage parameter $|\rho|$ as well as on $\mathbb{E}_{\mathbb{Q}}[Z_\lambda]$. Intuitively, if the currently observed VIX index is small, the current market's expectation about the volatility within the next month precludes too many large jumps in σ_t (modeled via large $\mathbb{E}_{\mathbb{Q}}[Z_\lambda]$) as well as too large downward jumps in S (modeled via large $|\rho|$). Only if the restriction (4) is in place, it is possible to explain the currently observed VIX index via the model, when choosing σ_0 according to

$$\sigma_0 = \sqrt{\frac{\text{VIX}_0^2 - c_V}{b_V}}.$$

The model is analytically tractable in the sense that European claims on both S_t and VIX_t can be priced via efficient Fourier pricing methods. This is because the characteristic functions of both $\log(S_t)$ and σ_t^2 are known in closed form. For detailed derivations we refer to Nicolato, Venardos (2003) (for calls on S) and Arai (2019) (for calls on VIX), but we present the main formulas here. To this end, we denote $\hat{u} := \sup\{u \in \mathbb{R} : -\Psi_{\nu_Z}(-u) < \infty\} \geq 0$, fix a maturity $u > 0$, and assume that

$$\int_{\mathbb{R}} \frac{|\phi_{u|t}(v - i\,\alpha)|}{1+|v|}\,\mathrm{d}v < \infty \qquad (5)$$

for any $t \in [0,u]$ and $\alpha \in (0,\hat{u})$. Then, taking the approximation (3) for granted as an equality, for arbitrary such t and α we obtain

$$\mathbb{E}_{\mathbb{Q}}\big[\big(\text{VIX}_u - K\big)_+ \,\big|\, \mathcal{F}_t\big] = \frac{1}{2\,\pi}\int_{\mathbb{R}} \hat{g}(v,\alpha;K)\,\phi_{u|t}(-v - i\,\alpha)\mathrm{d}v, \qquad (6)$$

where the Fourier transform of the payoff $\hat{g}$ and the characteristic function $\phi_{u|t}$ of σ_u^2 given σ_t^2 are known in closed form, and given by

$$\hat{g}(v,\alpha;K) = \frac{e^{-\frac{(i\,v-\alpha)\,c_V}{b_V}}\,\sqrt{b_V}}{(-i\,v+\alpha)^{\frac{3}{2}}}\int_{K\sqrt{\frac{-i\,v+\alpha}{b_V}}}^{\infty} e^{-x^2}\,\mathrm{d}x,$$

$$\phi_{u|t}(\zeta) = e^{i\,\zeta\,e^{-\lambda\,(u-t)}\,\sigma_t^2}\,\exp\Big\{-\int_t^u \Psi_{\nu_Z}\Big(-i\,\zeta\,e^{-\lambda\,(u-s)}\Big)\,\mathrm{d}s\Big\}.$$

In order to implement these formulas efficiently, a few remarks are helpful:

- The so-called complementary error function $2/\sqrt{\pi}\,\int_x^\infty \exp(-x^2)\,\mathrm{d}x$ in $\hat{g}$ is pre-implemented in most software packages as the `erfc`-function, also for complex arguments, as required here.
- The Laplace exponent Ψ_{ν_Z} has a convenient closed form expression for typical models that are applied in practice. In the Appendix in Section 5.2 we provide the respective expressions for the so-called IG-OU model, for which the technical condition (5) is shown to be satisfied in (Arai, 2019, Example 3.5).
- The (approximate) evaluation of the integral $\int_{\mathbb{R}}$ in (6) can be accomplished quite efficiently via the Fast Fourier Transform (FFT) algorithm.

Regarding calls on S itself, required is the characteristic function $\Phi_{u|t}$ of $\log(S_u)$ given $\big(\log(S_t),\sigma_t^2\big)$, which is derived in Nicolato, Venardos (2003) and given by

$$\Phi_{u|t}(\zeta) = \exp\Big\{i\,\zeta\,\big(\log(S_t)+\Psi_{\nu_Z}(-\rho)\,(u-t)\big) - \zeta^2\,\frac{1-e^{-\lambda\,(u-t)}}{2\,\lambda}\,\sigma_t^2 - \int_t^u \lambda\,\Psi_{\nu_Z}\Big(-\rho\,i\,\zeta+\frac{1}{2}\,\zeta^2\,\frac{1-e^{-\lambda\,(u-s)}}{\lambda}\Big)\mathrm{d}s\Big\}.$$

Given this expression, European call prices on S_u can be computed according to the formula

$$\mathbb{E}_{\mathbb{Q}}[(S_u-K)_+\,|\,\mathcal{F}_t] = \frac{e^{-\alpha\,\log(K)}}{\pi}\times \int_0^\infty \mathrm{Re}\Big[\Phi_{u|t}(-v-i\,(\alpha+1))\,\hat{h}(v,\alpha;K)\Big]\,\mathrm{d}v,$$

where the Fourier transform of the payoff $\hat{h}(v,\alpha;K)$ is given by

$$\hat{h}(v,\alpha;K) = \frac{e^{i\,v\,\log(K)}}{(\alpha-i\,v)\,(\alpha-i\,v+1)},$$

and required is the assumption that $\mathbb{E}_{\mathbb{Q}}[S_T^{\alpha+1}]<\infty$.

5 Appendix

5.1 Model-free delta and gamma of logarithmic claim

We assume that $S=\{S_t\}_{t\geq 0}$ is a Markovian stochastic process under $\mathbb{Q}$, satisfying for $0\leq t\leq T$ that

$$S_T = S_t\,e^{r\,(T-t)}\,(M_T-M_t),$$

where $M = \{M_t\}_{t \geq 0}$ is a martingale (w.r.t. filtration generated by S) with $M_0 = 1$ and $M_T - M_t$ independent of S_t. This assumption is model-free in the sense that it leaves unspecified the choice of the martingale M, which can be quite arbitrary. For instance, it is an assumption satisfied by common diffusion and Lévy models. We observe that

$$\begin{aligned} V(t, S_t) &= \mathbb{E}_{\mathbb{Q}}[\log(S_T) \,|\, S_t] \\ &= \log(S_t) + \underbrace{r\,(T-t) + \mathbb{E}_{\mathbb{Q}}[\log(M_T - M_t)]}_{=:f(t)} \end{aligned}$$

for some model-dependent function f of time t. Hence, $V_s(t, S_t) = 1/S_t$ and $V_{ss}(t, S_t) = -1/S_t^2$, independently of f, i.e. independent of the model for M.

5.2 The IG-OU process

The Inverse Gaussian (IG) distribution is a two-parametric infinitely divisible probability distribution on $(0, \infty)$, which we denote by IG(δ, γ). Its associated Bernstein function is given by

$$\Psi_{\nu_{IG}}(x) = \delta\left(\sqrt{\gamma^2 + 2\,x} - \gamma\right), \quad x \geq 0,$$

for parameters $\delta, \gamma > 0$. It is known to be self-decomposable, which implies that there exists a stationary process $\sigma_t^2 \sim$ IG(δ, γ), $t \geq 0$, which has the representation

$$\sigma_t^2 = e^{-\lambda\,t}\,\sigma_0^2 + e^{-\lambda\,t}\int_0^t e^{\lambda\,s}\,\mathrm{d}Z_{\lambda\,s}, \quad t \geq 0,$$

for some Lévy subordinator Z, which is called the background driving Lévy subordinator. The Bernstein function of $\{Z_{\lambda\,t}\}_{t \geq 0}$ is known to be given by

$$\Psi_{\nu_Z}(x) = \frac{\delta\,x}{\sqrt{\gamma^2 + 2\,x}}, \quad x \geq 0,$$

with Lévy measure ν_Z given by

$$\nu_Z(\mathrm{d}x) = \frac{\delta}{2\sqrt{2\,\pi}}\,x^{-\frac{3}{2}}\,(1 + \gamma^2\,x)\,e^{-\frac{1}{2}\gamma^2\,x}\,\mathrm{d}x.$$

An exact simulation algorithm for paths of the process σ_t^2 along a grid is provided in Zhang, Zhang (2007). In the case $\rho = 0$ (no leverage effect) this can be used to simulate exactly paths of S as well. In the case $\rho < 0$, however, required is a joint simulation together with the process $\{Z_{\lambda\,t}\}_{t \geq 0}$. For practical purposes, it might be sufficient to apply a standard Euler scheme based on the recursions

$$\begin{aligned} \mathrm{d}X_t &= \big(r + \Psi_{\nu_Z}(-\rho)\big)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t + \rho\,\mathrm{d}Z_{\lambda\,t}, \\ \mathrm{d}\sigma_t^2 &= -\lambda\,\sigma_t^2\,\mathrm{d}t + \mathrm{d}Z_{\lambda\,t}, \end{aligned}$$

where $X_t = \log(S_t)$. In order to simulate the increments of $\{Z_{\lambda\,t}\}_{t \geq 0}$ we observe that

$$\Psi_{\nu_Z}(x) = \underbrace{\frac{\delta}{2}\left(\sqrt{\gamma^2 + 2\,x} - \gamma\right)}_{=:\Psi_1(x)} + \underbrace{\frac{\gamma\,\delta}{2}\left(1 - \frac{1}{\sqrt{1 + \frac{2}{\gamma^2}\,x}}\right)}_{=\Psi_2(x)},$$

where Ψ_1 and Ψ_2 are recognized as well-known Bernstein functions. To wit, we obtain $Z_{\lambda t} \sim Z_t^{(1)} + Z_t^{(2)}$, where $Z^{(1)}, Z^{(2)}$ are independent, $Z^{(1)}$ is an Inverse Gaussian subordinator with parameters $(\delta/2, \gamma)$ and $Z^{(2)}$ a compound Poisson subordinator with intensity $\gamma\,\delta/2$ and $\Gamma(1/2, \gamma^2/2)$-distributed jumps, where we refer to the notation/ parameterization in (Mai, Scherer, 2017, p. 309 and p. 2) for the Inverse Gaussian subordinator and the Gamma distribution, respectively.

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