XAIA INVESTMENT

Correction of "Portfolio opimization for credit-risky assets under Marshall–Olkin dependence"

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Date: January 28, 2021

Abstract The author's article Mai (2020) solves portfolio selection for credit-risky assets under Marshall–Olkin dependence based on the concept of power/logarithmic utility maximization. This article contains three crucial mistakes. First, an analogy with Markowitz was presented in a didactic manner that was unfortunate, caused by a confusion of regular and stochastic exponential. Second, a leverage constraint was mis-specified, which resulted in an error in situations with short-selling allowed. Third, there was a numerical error in an analysis that pointed out how higher-order dependence properties beyond first and second moments can have a significant impact on optimal portfolios. Luckily, all three mistakes can be corrected without altering the essential contributions and statements of the original article, which is the content of the present correction note.

1 Introduction

Mai (2020) considered a d-variate asset price process $\boldsymbol{B}(t) = \big(B_1(t), \ldots, B_d(t)\big)$, where each B_i has the multiplicative decomposition $B_i(t) = B_i(0)\, e^{\mu_i\, t}\, B_i^{(BS)}(t)\, B_i^{(MO)}(t)$. Regarding interpretations, $B_i(0)$ equals the current asset value, μ_i denotes the average rate of return, and $B_i^{(BS)}$ and $B_i^{(MO)}$ are two independent martingales. The latter are given as

$$B_i^{(BS)}(t) = e^{-\frac{\sigma_i^2}{2}\, t + \sigma_i\, W_i(t)},$$
$$B_i^{(MO)}(t) = e^{(1-\kappa_i)\,\Lambda_i\, t}\,\big(1_{\{\tau_i > t\}} + \kappa_i\, 1_{\{\tau_i \leq t\}}\big),$$

with volatility parameters $\sigma_i \geq 0$ and possibly correlated Brownian motions W_i, recovery rate parameters $\kappa_i \in [0,1]$ and possibly dependent exponential random variables τ_i with rates $\Lambda_i \geq 0$. The special case $\Lambda_i = 0$ for all i boils down to the multivariate Black-Scholes model, whereas the case $\sigma_i = 0$ for all i boils down to a model focusing exclusively on the risk that the assets $i \in \{1, \ldots, d\}$ default at times τ_i. Mai (2020) further assumes that the random vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)$ has a *Marshall–Olkin distribution* and justifies this modeling choice as a convenient trade-off between realism and tractability. Quickly recapped,

$$\tau_i \sim \min\{E_I \,:\, i \in I\}, \quad i = 1, \ldots, d,$$

where for each non-empty subset $I \subset \{1, \ldots, d\}$ the random variable E_I is exponential with rate $\lambda_I \geq 0$, and all E_I are mutually independent. Intuitively, E_I represents the arrival time point of an economy shock that affects all assets with indices $i \in I$, and for arbitrary subset I the random variable $\min\{\tau_i \,:\, i \in I\}$ is exponential with rate equal to $\Lambda_I := \sum_{J\,:\,J \cap I \neq \emptyset} \lambda_J$. This renders

the Marshall–Olkin distribution a natural multivariate generalization of the univariate exponential distribution and, in particular, makes clear that each τ_i is exponential with rate $\Lambda_i := \Lambda_{\{i\}}$.
As a major result, Mai (2020) proved that keeping the portfolio allocations identically constant until the first observed asset default at $\tau_{[1]} := \min\{\tau_1, \ldots, \tau_d\}$ is optimal in the sense that this trading strategy maximizes the expected power/logarithmic utility of terminal wealth. Recall that with a risk aversion parameter $p < 1$ the respective utility function is defined as $U_p(x) = (x^p - 1)/p$, which for $p \neq 0$ equals $U_0(x) = \log(x)$. Furthermore, the optimal portfolio allocation $\boldsymbol{x}_*$ equals the maximizer of the concave function

$$\begin{aligned} g_{p,d}(\boldsymbol{x}) &= \langle \boldsymbol{x}, \boldsymbol{\mu} \rangle - \frac{1-p}{2} \langle \boldsymbol{x}, \Sigma\, \boldsymbol{x} \rangle \\ &\quad + \sum_{i=1}^{d} \Lambda_i\,(1-\kappa_i)\,x_i + \frac{1}{\mathbb{E}[\tau_{[1]}]}\,\mathbb{E}\Big[U_p\Big(1 - \sum_{i \in S}(1-\kappa_i)\,x_i\Big)\Big], \end{aligned} \tag{1}$$

where Σ is the covariance matrix of $\big(\sigma_1 W_1(1), \ldots, \sigma_d W_d(1)\big)$, and S denotes the random subset of $\{1, \ldots, d\}$ indexing all assets that default at $\tau_{[1]}$. The maximization of $g_{p,d}$ amounts to a trade-off between maximizing the expected portfolio return $\langle \boldsymbol{x}, \boldsymbol{\mu} \rangle$ and minimizing two risk measures. While the portfolio return variance $\langle \boldsymbol{x}, \Sigma\, \boldsymbol{x} \rangle$ is a risk measure that corresponds to the Black-Scholes (BS) part of the model, the expression in the second line is associated with the Marshall–Olkin (MO) part of the model. In particular, the last expectation value can be viewed as the negative of a particular risk measure that is taylor-made to the MO part of the model.
The organization of the remaining article is as follows. Section 2 corrects a mistake due to a confusing mix between different ways of thinking in an equity paradigm and a credit paradigm. Section 3 corrects a mistake regarding a leverage mistake when short-selling is admissible. Section 4 corrects an analysis of (Mai, 2020, Section 4) that shows how higher-order dependence properties beyond first and second moments can have significant implications for portfolio optimality.

2 An important didactic correction

The first important mistake in Mai (2020) relies on a confusion of regular and stochastic exponential. The stochastic process $\exp(\mu_i\, t)\, B_i^{(MO)}(t) = \mathcal{E}(R_i^{(MO)})_t$ equals the stochastic exponential of the log-return process

$$R_i^{(MO)}(t) = \eta_i\, t\, 1_{\{\tau_i > t\}} + \big(\eta_i\, \tau_i + \kappa_i - 1\big)\, 1_{\{\tau_i \leq t\}},$$

where $\eta_i := \mu_i + \Lambda_i\,(1-\kappa_i)$. In the introduction of Mai (2020) the expression $\kappa_i - 1$ in the last formula was falsely replaced by $\log(\kappa_i)$ and also η_i was replaced by μ_i, which was due to a confusion with the regular logarithm of $B_i^{(MO)}(t)$. As a result, the term $\sum_{i=1}^{d} \Lambda_i\,(1-\kappa_i)\,x_i$ in (1) was missing in Mai (2020). The author's confusion actually relied on an important difference between the common equity paradigm (traditionally modeled by BS) and the credit paradigm (modeled by MO here). We find it educational to point this out in the following.

In the equity paradigm a single asset is described by two core metrics, the expected return μ_i and the standard deviation σ_i as a measurement for symmetric deviation from the average. This is naturally a unimodal return distribution. In contrast, in the credit paradigm a single asset return is described by the three natural core metrics $(\eta_i, \Lambda_i, \kappa_i)$. It is naturally a bi-modal return distribution, where the parameter Λ_i controls the probability between two extreme outcomes ("survival" versus "default") and the pair (η_i, κ_i) models the returns in the two extreme cases. From the point of view of a credit investor these three core metrics are the natural model parameters, whereas the average return μ_i is a rather unnatural quantity. For instance, when B_i describes a bond, then a recovery rate assumption κ_i as well as a credit spread estimate Λ_i are natural quantities for investors, and a reasonable choice for η_i would be the bond yield, also a common quantity to credit investors. In contrast, an expected return μ_i in practice is derived from these three quantities via the formula $\mu_i = \eta_i - \Lambda_i\,(1-\kappa_i)$ rather than modeled, unlike in the equity paradigm. In fact, in typical situations μ_i is not even in the support of $R_i^{(MO)}(1)$
Summarizing, in the pure MO model it is more natural to omit the parameter μ_i and instead replace it by $\eta_i - \Lambda_i\,(1-\kappa_i)$ in all formulas, introducing the more natural yield parameter η_i.

3 Leverage constraint in case of short-selling

The domain of $g_{p,d}$ needs to be restricted by a leverage constraint. Intuitively, if all assets $i \in S$ default at $\tau_{[1]}$, then this induces the proportional loss $\sum_{i\in S}(1-\kappa_i)\,x_i$. In order not to be bankrupt the investor should thus make sure that this proportional loss remains bounded from above by one. Denote by $\mathcal{J}$ the collection of subsets of $\{1,\ldots,d\}$ that have a positive probability to be potential outcomes of S, i.e. $\mathcal{J} = \{I\,:\,\lambda_I > 0\}$. Then $\boldsymbol{x}$ needs to be chosen from the domain

$$\mathcal{C}^0 := \bigcap_{I\in\mathcal{J}} H_I, \quad H_I := \Big\{\boldsymbol{x}\in\mathbb{R}^d\,:\,\sum_{i\in I}(1-\kappa_i)\,x_i \le 1\Big\}.$$

In Mai (2020) it was falsely claimed that $\mathcal{C}^0 = H_{\{1,\ldots,d\}}$. The reason this mistake was overlooked is that the author designed the model for situations without short-selling allowed. If one imposes a short-selling ban, meaning that one imposes the additional constraint $\boldsymbol{x} \ge 0$ on admissible portfolios, then consideration of the subset $\{1,\ldots,d\}$ is sufficient, since (assuming $\{1,\ldots,d\}\in\mathcal{J}$)

$$[\mathbf{0},\infty)\cap\mathcal{C}^0 = [\mathbf{0},\infty)\cap H_{\{1,\ldots,d\}}.$$

However, with short-selling allowed this is no longer true. For instance, in the case $d=2$ the proportional portfolio allocation $(x_1,x_2) = \big(-1/(1-\kappa_1), 2/(1-\kappa_2)\big)$ is certainly always in $H_{\{1,2\}}$, but never in $H_{\{2\}}$. Intuitively, if asset 2 defaults isolated, i.e. $S = \{2\}$, then the investor looses more wealth than he actually has, which is inadmissible.

4 Correction of the analysis in Section 4 of Mai (2020)

(Mai, 2020, Section 4) is dedicated to prove by a deliberately chosen example that two different Marshall–Olkin distributions for the law of $\boldsymbol{\tau}$ can imply two extremely different optimal portfolios, despite the fact that both distributions imply the exactly identical distributions for all marginal bivariate pairs (τ_i, τ_j). This finding is of theoretical interest, since it is a feature that sets the MO paradigm apart from the BS paradigm, in which bivariate marginal distributions fully determine the multivariate normal return distribution. Unfortunately, the implementation of (Mai, 2020, Section 4) contains an error implying that the plots for families (1), (4), (5), and (6) in (Mai, 2020, Figure 3) are all wrong. If this error is corrected, these plots look almost exactly identical to the plots for families (2) and (3), which themselves also look almost identical. Given this, the example does not achieve its desired goal of demonstrating differences between the six considered different model specifications.

Luckily, the idea of the analysis in (Mai, 2020, Section 4) was correct and the example can easily be modified in order to accomplish its goal. To this end, we found out that the plots for all six families are almost identical, because the BS part dominated the MO part of the return distribution in the constructed example by far. This was an unintended and unfortunate parameter choice. Consequently, removing the BS part makes the desired difference visible. We now formally repeat and correct this analysis. Formally, the Marshall–Olkin distribution of $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)$ in the example was constructed by considering the first d members of the infinite exchangeable sequence defined via

$$\tau_i := \inf\{t > 0 \,:\, L_t > \epsilon_i\}, \quad i \geq 1,$$

where $\epsilon_1, \epsilon_2, \ldots$ is a sequence of iid standard exponential random variables and $L = \{L_t\}_{t\geq 0}$ denotes an independent Lévy subordinator with Laplace exponent Ψ. We furthermore assume that $\kappa_1 = \kappa_2 = \ldots = 0$ and $\eta_1 = \eta_2 = \ldots = \eta = 0.04$, where $\eta = \mu + \Lambda$ denotes the more natural yield parameter introduced in Section 2. Since the probability law of $\{\tau_i\}_{i\geq 1}$ is invariant under re-ordering of the τ_i by construction, and yield parameters as well as recovery rates are also identical, there is no preference for a specific asset and an optimizer $\boldsymbol{x}_*$ of $g_{p,d}$ is necessarily of the form $\boldsymbol{x}_* = (c_{p,d}, \ldots, c_{p,d})/d$ for some constant $c_{p,d}$. The only question is: "how large is $c_{p,d}$, and thus the cash ratio?" The parameters λ_I for the Marshall–Olkin distribution of $(\tau_1, \ldots, \tau_d)$ depend on Ψ, and we consider six different specifications for Ψ, all of the form $\Psi(x) = \Lambda\, \Psi_\theta(x)$ for a parameter $\Lambda = 0.03$ and a one-parametric Laplace exponent Ψ_θ with the constraint $\Psi_\theta(1) = 1$, and drift b_θ as well as Lévy measure ν_θ. The parameter Λ equals the exponential rate of the τ_i and the parameter θ controls the copula (i.e. dependence structure). The six families we consider are depicted in (Mai, 2020, Table 1), with families (2) and (4) being compound Poisson subordinators, families (1) and (5) being infinitely active subordinators, and families (3) and (6) compound Poisson subordinators with additional drift. For all six families we choose the parameter θ such that $2 - \Psi_\theta(2) = 0.3$ is identical, which implies that the probability distribution of (τ_i, τ_j) for $i \neq j$ is identical in all six model specifications. But the probability distribution of $(\tau_1, \ldots, \tau_d)$ for $d > 2$ is truly different. Figure 1 visualizes the probability distribution of the number $|S|$ of defaults at time $\tau_{[1]}$ for $d = 20$ in all six models.

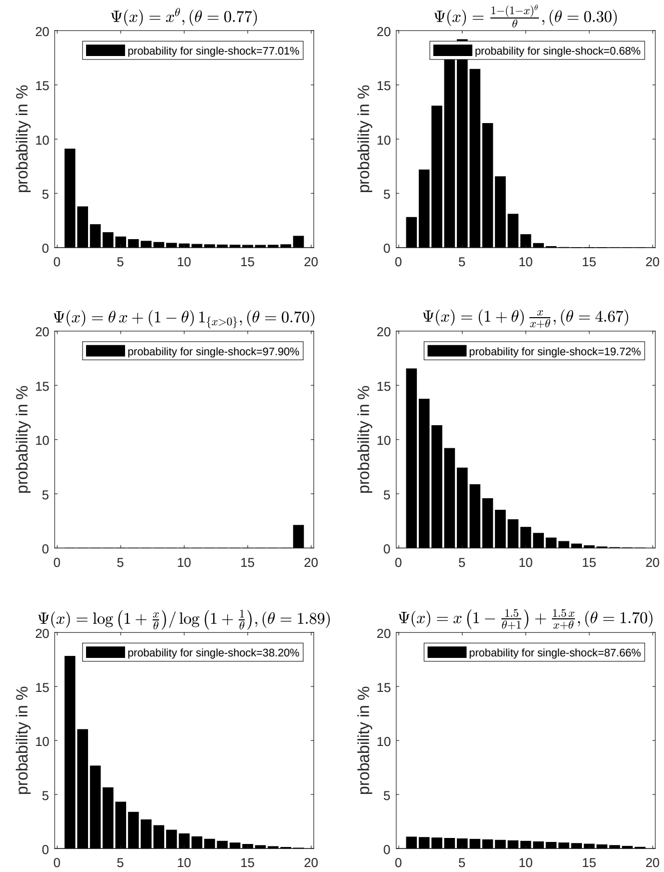

Fig. 1: The probability distribution $p_n := \mathbb{P}(|S| = n)$ for $n \in [d]$ with $d = 20$, for the six different families of (Mai, 2020, Table 1). The probability $\mathbb{P}(|S| = 1)$ of a single shock is depicted separately as a legend entry. The respective Laplace exponents are depicted, and the parameters θ are chosen such that $2 - \Psi_\theta(2) = 0.3$ in all six examples, which implies that the probability law of (τ_i, τ_j) is identical in all examples for $i \neq j$.

The optimal constant $c_{p,d}$ is found by maximizing the function (with a slight abuse of notation)

$$g_{p,d}(c) := g_{p,d}\Big(\big(\frac{c}{d}, \ldots, \frac{c}{d}\big)\Big) = c\,\eta + \sum_{k=1}^{d} \binom{d}{k}\, \lambda_{\{1,\ldots,k\}}\, U_p\Big(1 - c\,\frac{k}{d}\Big)$$
$$= c\,\eta + \Lambda_{[d]}\, \mathbb{E}\Big[U_p\Big(1 - c\,\frac{|S|}{d}\Big)\Big].$$

Notice that in comparison to the definition of $g_{p,d}$ in (1) here we aggregate the two sums $\langle \boldsymbol{x}, \boldsymbol{\mu} \rangle$ and $\sum_{i=1}^{d} \Lambda_i\,(1-\kappa_i)\,x_i$, yielding $\mu\,c + \Lambda\,c = \eta\,c$. As $d \to \infty$, the function $g_{p,d}$ tends to

$$g_{p,\infty}(c) = \begin{cases} (\eta - \Lambda\,b_\theta)\,c - \frac{\Lambda}{p} \int_{(0,\infty]} 1 - \big(1 - c\,(1 - e^{-x})\big)^p\, \nu_\theta(\mathrm{d}x) \\ (\eta - \Lambda\,b_\theta)\,c + \Lambda \int_{(0,\infty]} \log\big(1 - c\,(1 - e^{-x})\big)\, \nu_\theta(\mathrm{d}x) \end{cases},$$

where the first case is valid for $p \neq 0$ (power utility) and the second case for $p = 0$ (logarithmic utility). Figure 2 visualizes the functions $g_{0,d}$ for $d \in \{2, 5, 10, 20, \infty\}$ together with the associated optimal values $c_{0,d}$.

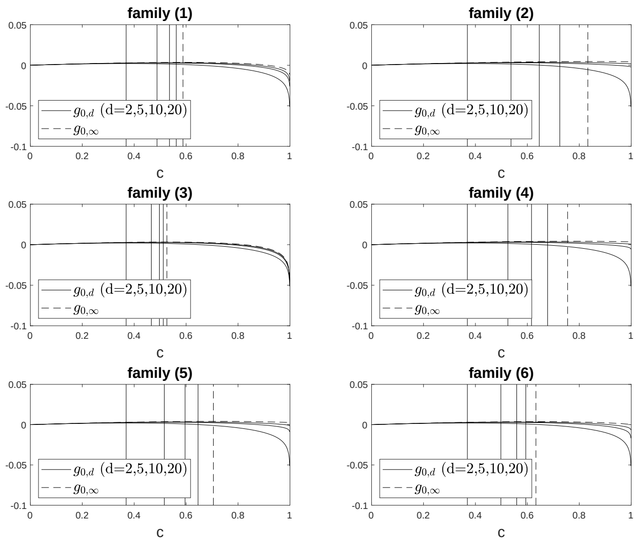

Fig. 2: The functions $g_{0,d}$ for $d \in \{2, 5, 10, 20, \infty\}$ for the same six parametric families as given in (Mai, 2020, Table 1), each time with θ such that $2 - \Psi_\theta(2) = 0.3$. Further parameters are chosen as $\eta = 0.04$ and $\Lambda = 0.03$. Vertical lines visualize the optimal investment ratio c. The dotted lines correspond to $g_{0,\infty}$, whereas solid lines correspond to $g_{0,d}$ for $d < \infty$.

The following observations are important in these plots, and prove that different higher-order dependence specifications can have a significant impact on optimal portfolios:

- The optimal value $c_{0,2}$ is identical for all six families. This is due to the fact that (τ_1, τ_2) has the identical distribution in all considered models, but only the laws of $(\tau_1, \ldots, \tau_d)$ for $d > 2$ are truly different.

- In all families the optimal value $c_{0,d}$ increases in the dimension d. This means that with increasing dimension it is optimal to increase the investment ratio, i.e. the invested proportion of wealth. Intuitively, this can be explained with a *diversification benefit*. A larger asset universe leaves more opportunity to diversify the risk among many assets, so that the risk-return profile improves.

- For those families with large probability of economy shocks affecting many assets, for instance with large probability $\mathbb{P}(S = [d]) = \mathbb{P}(|S| = d)$, the aforementioned diversification benefit is not as huge as for the families in which these probabilities are smaller. For instance, family (3) has the largest intensity of a "kill-all" shock $\lambda_{[d]} = (1 - \theta)\,\Lambda$, irrespective of the dimension d. This risk cannot be reduced by means of diversification when considering more assets. Consequently, $c_{0,d}$

does not increase too much in d. In contrast, for family (2) the diversification benefit is huge and $c_{0,\infty} \gg c_{0,2}$, the reason being very small probabilities for "kill-all" shocks. Intuitively, the other four considered families are somewhere in between these two most extreme cases.

Acknowledgements The author gratefully thanks Ales Cerny and Alexandra Blagoeva for pointing out to him some mistakes in the original article Mai (2020). The corrections of these mistakes is what led to this correction note.

References

J.-F. Mai, Portfolio opimization for credit-risky assets under Marshall–Olkin dependence, *Applied Mathematical Finance* **26** (2020) 598–618.