XAIA INVESTMENT

ON FOREIGN CURRENCY EQUITY DRIFT RATES

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Abstract The post-crisis market standard bootstrap of a risk-free rate for a foreign currency typically ensures that interest rate swaps in domestic and foreign currency, as well as cross currency swaps between the two currencies, trade at zero, see, e.g., Fujii et al. (2010). Whereas the pre-crisis risk-free rate for the foreign currency, which is bootstrapped from only interest rate swaps in the foreign currency, is no longer used for discounting foreign future cash flows, it is still required for the computation of coupon payments resulting from a foreign currency floating rate bond. The present note points out that also for the pricing of an equity derivative based on a foreign currency stock it is reasonable to work with both the post-crisis and the pre-crisis risk-free rates. While the post-crisis risk-free rate is required to discount future cash flows from the derivative, the pre-crisis risk-free rate is required to model the drift of the stock price process. If the drift of the stock price process is (poorly) modeled using the post-crisis risk-free rate, the equity model is inconsistent with the cash market model. For example, it implies non-zero values for an equity forward contract.

1 Bootstrapping risk-free cash accounts

For the sake of a simplified notation we assume that all involved interest rate curves are deterministic, and we assume that the domestic currency is the €, while the foreign currency is the \$. For $x, y \in \{\text{€}, \$\}$, we denote by $r_x^y(t)$ the short rate that is used by an x-based investor for discounting future cash flows in y, i.e. a future y-cash flow at time $t > 0$ is discounted back into today by the x-based investor using the discount factor $\exp(-\int_0^t r_x^y(s)\,\mathrm{d}s)$. For us as a €-based investor it is common market practice (post- and pre-crisis) to bootstrap the rate $r_\text{€}^\text{€}(t)$ from domestic interest rate swaps. However, according to Fujii et al. (2010), the post-crisis method to retrieve the rate $r_\text{€}^\$(t)$ does not only involve foreign currency interest rate swaps, but additionally domestic interest rate swaps and cross currency swaps between \$ and €. This is necesary in order to explain the fair market prices of cross currency swaps. The latter are quoted in terms of the so-called cross currency basis swap spread that trades significantly away from zero since the collapse of Lehman in 2008, see Figure 1. Before the crisis, it has been market standard to ignore the cross currency basis swap spread (or assume that it equals zero), which implies the equalities $r_\text{€}^\$(t) = r_\$^\$(t)$ and $r_\$^\text{€}(t) = r_\text{€}^\text{€}(t)$. In other words, there was one rate for € and one for \$, but they were identical for both the €-based and the \$-based investor. In Moreni, Pallavicini (2015) and the references

cited therein one can find some economic explanations for the fact that the cross-currency basis swap spread has become so negative after 2008. One of the most intuitive arguments is the market's perception that the European financial system is more risky than the US financial system. Following the logic of this argument, the cross-currency basis swap spread is considered a premium the $-provider in a cross currency swap earns for taking the risk of a loss incurred as a result of the breakdown of the European financial system (in which case the cross currency swap counterparty defaults and simultaneously the €-collateral depreciates).

The fact that €-based investors and $-based investors use different rates for discounting cash flows is typically justified by an interpretation in terms of funding costs. The negative cross currency basis swap spread in Figure 1 implies that there is higher demand for $ than for €. If we as €-based investors need to fund $, this comes at a cost. Conversely, having $ cash corresponds to a funding benefit[1]. During the bootstrapping procedure, non-zero values of cross currency basis swap spreads result in a difference $x(t)$ between the rates $r^\$_{\text{€}}(t)$ and $r^\$_\$(t)$, i.e. $r^\$_{\text{€}}(t) = r^\$_\$(t) - x(t)$. The size of $x(t)$ is approximately given by the prevailing cross currency basis swap spread that is quoted for a cross currency swap with maturity t. In the current market situation, $x(t) < 0$ for all t, see the left plot in Figure 2. Since the model idealistically assumes that cash can be invested at and borrowed at the same risk-free rate, negative $x(t)$ conversely means that the €-based investor's risk-free $-bank account earns more than the risk-free $-bank account of the $-based investor. This counterintuitive assumption must be considered a trade-off for keeping the model simple by not having to introduce two different rates for negative and positive balances on the risk-free bank accounts[2]. Whereas by construction $- and €-interest rate swaps as well as cross currency swaps all have the value zero for both $-based and €-based investors in the model, a $-floating rate bond is worth par to the $-based investor but trades below par for the €-based investor. This reflects the idea that in order to buy this $-floating rate bond (for one $) the €-based investor must fund the initial buy price at a cost. Consequently, in this model world there are two different prices for the $-floating rate bond. This implies that at any price in between these two values the $-guy would buy the bond from the €-guy, who is happy to sell at that price, but a trade the other way round would not take place (i.e. the €-guy would not buy from the $-guy).

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[1]Alternatively, one might also think of a negative cross currency basis swap spread as a compensation for taking the risk of investing in €. With this logic, it plays the role of a CVA-adjustment to the $-based investor and a DVA-adjustment to the €-based investor. From point of view of the €-based investor, Morini, Prampolini (2011) explain how the funding interpretation and the DVA-interpretation can be pulled together.

[2]Although desirable from a theoretical perspective, such a model introduces issues that complicate the pricing for even the simplest type of contracts massively, see, e.g. Fries (2010); Pallavicini et al. (2012).

Fig. 1: Historical evolution of the cross currency basis swap spread (in bps) between EUR and USD for swaps with five year maturity (Source: Bloomberg).

2 Check-marked cash, what about equity?

Now assume that we as €-based investor buy an uncollateralized equity derivative on an underlying stock price process $\{S_t\}_{t\geq 0}$ that is denoted in \$. Say it is a European-style derivative with payoff $f(S_T)$ at a future time point T. According to arbitrage pricing theory, the value of this derivative for us equals

$$V_{\mathbb{€}} = e^{-\int_0^T r_{\mathbb{€}}^{\$}(t)\,\mathrm{d}t}\,\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big], \qquad (1)$$

because the payoff $f(S_T)$ consitutes a future cash flow in \$, which we need to discount at the rate $r_{\mathbb{€}}^{\$}(t)$. How should we model the dynamics of $\{S_t\}_{t\geq 0}$ under a risk-neutral measure $\mathbb{Q}$? On the one hand, we could model the stock price with drift $r_{\mathbb{€}}^{\$}(t)$ in order to be consistent with pre-crisis methodology, but this would imply that €- and \$-based investors work not only with different numeraires but also with different risk-neutral measures $\mathbb{Q}$. On the other hand, for the \$-based investor the drift of the stock price process is clearly given by $r_{\$}^{\$}(t)$. If we use the same drift as €-based investor, then this would mean that the formula for $V_{\mathbb{€}}$ includes two different rates $r_{\mathbb{€}}^{\$}(t)$ and $r_{\$}^{\$}(t)$, which we are not used to according to pre-crisis market practice.

We argue that the second choice is the one consistent with the aforementioned funding interpretation. To this end, consider an equity forward contract with payoff $f(S_T) = S_T - K$ for a strike price K. This contract is available in the marketplace at zero cost, which means that $K = \mathbb{E}^{\mathbb{Q}}[S_T]$ is observable in the marketplace. Consequently, both \$-based and €-based investors should calibrate their models in such a way that they agree on $\mathbb{E}^{\mathbb{Q}}[S_T]$, i.e. on the drift of the stock under $\mathbb{Q}$. Since this drift obviously equals $r_{\$}^{\$}(t)$ for the \$-based investor, the case is decided in favor of $r_{\$}^{\$}(t)$, i.e. the formula for $V_{\mathbb{€}}$ requires the knowledge of both rates $r_{\mathbb{€}}^{\$}(t)$ and $r_{\$}^{\$}(t)$. But is the postulate of a law of one price for equity forward contracts consistent with the aforementioned funding interpretation? Recall that the model assumes that the risk-free \$-bank account of the €-based investor earns the rate $r_{\$}^{\$}(t) - x(t)$, which for negative $x(t)$ exceeds the earnings of the

risk-free \$-bank account of the \$-based investor by $-x(t)$. Consequently, it is only consistent to also assume that ownership of a \$-stock generates a continuous "dividend yield" $-x(t)$, but only to the €-based investor. This continuous dividend yield might also be interpreted as a repo margin, i.e. the €-based investor (but not the \$-based investor!) is able to lend his \$-stock continuously, earning the repo margin $-x(t)$. Consequently, whereas the $r_{€}^{\$}(t)$-discounted stock price process is not a martingale under $\mathbb{Q}$, the $r_{€}^{\$}(t)$-discounted wealth process of a portfolio holding the stock is a martingale for the €-based investor, because continuous stock lending generates value for the portfolio additional to the stock price.

Remark 2.1 (Collateralized equity derivatives)
Now we assume the equity derivative is fully collateralized with an idealized continuous two-way zero threshold CSA. The latter assumption eliminates counterparty credit risk, so that we can concentrate on the funding interpretation. It is assumed that the collateral may be re-hypothecated and must be paid back at a contractually specified rate $r_C(t)$. It is well-known[3] that in this case formula (1) changes to

$$V_{€} = \begin{cases} e^{-\int_0^T r_C(t)\,\mathrm{d}t}\, \mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big] & \text{, if collateral in \$,} \\ e^{-\int_0^T r_{€}^{\$}(t)-r_{€}^{€}(t)+r_C(t)\,\mathrm{d}t}\, \mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big] & \text{, if collateral in €,} \end{cases} \tag{2}$$

However, the question of the appropriate drift of the stock price process under $\mathbb{Q}$ is still unanswered. In our view, this question is decoupled from the presence or absence of collateralization. And the answer should be the same as in the uncollateralized case: modeling the drift of the stock price process as $r_{\$}^{\$}(t)$ is necessary if one desires equity forward contracts to have a price of zero.

Let us finally provide a realistic example of how severe the drift rate assumption affects the pricing of stock derivatives. On October 8, 2015, we consider call option data on the Dow Jones with maturity December 2017. Figure 2 shows implied volatilities for observed call option prices, which are computed under the assumption of a Geometric Brownian motion for the underlying. The computation of these implied volatilities is done in two different ways. Once we assume that the drift rate of the Dow Jones index equals $r_{\$}^{\$}(t)$, once the drift rate is assumed to equal $r_{€}^{\$}(t) = r_{\$}^{\$}(t) - x(t)$ (which is inappropriate according to the arguments provided earlier).
The risk-neutral expectations $\mathbb{E}^{\mathbb{Q}}[S_T]$ depend on the assumed drift rate. Since $r_{€}^{\$}(t)$ lies above $r_{\$}^{\$}(t)$, the (wrong) use of the rate $r_{€}^{\$}(t)$ overestimates this expectation, and hence call prices. Consequently, the implied volatilities of observed market prices are systematically underestimated. Figure 2 shows that the price differences range from almost zero (for out-of-the-money strikes)

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[3]See, e.g., Piterbarg (2010); Pallavicini et al. (2012); Moreni, Pallavicini (2015), who provide derivations. For the convenience of the reader, we provide a derivation in the present notation in the Appendix.
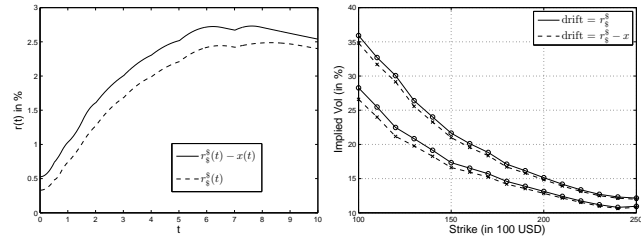
Fig. 2: Left: The short rate $r_\$^\$(t)$, bootstrapped from 3-month-tenor based $\$$ interest rate swaps, in comparison with the rate $r_{\text{€}}^\$(t) = r_\$^\$(t) - x(t)$, bootstrapped from 3-month-tenor based interest rate swaps in $\$$ and € as well as cross currency swaps. The difference between the rates constitutes the basis swap spread $x(t)$ (Date: October 8, 2015). Right: Implied volatilities for the observed option prices (bid and ask prices).

up to one vol point (for in-the-money strikes). For the sake of curiosity, Figure 3 shows the precisely same implied volatilities as Figure 2, the only difference being the involved interest rate term structures. The latter have now been bootstrapped on 30 December 2011, when the cross currency basis swap spread has been at an even wider level than now, cf. Figure 1. It is observed that $x(t)$ in this case is smaller than -1% for maturities up to two years. Consequently, the effect on call prices with maturity December 2017 is more dramatic, with price differences ranging up to 5 vol points.
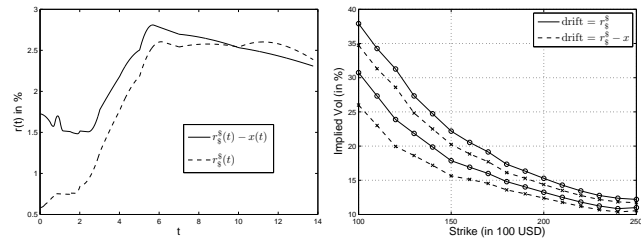

Fig. 3: Same data as in Figure 2, only that the involved interest rate term structures are bootstrapped on 30 December 2011, when the cross currency basis swap spread has been very negative on the short end of the curve, which is relevant here.

3 Summary   When using cross currency basis swap spread-adjusted discount factors in the sense of Fujii et al. (2010) for discounting future $\$$-cash flows, then we must be aware that we implicitly assume that (i) a €-based investor has access to a risk-free $\$$-bank account that earns a higher rate than the risk-free $\$$-bank account of the $\$$-based investor. That's why *"the €-guy wants the $\$$-cash more than the $\$$-guy"*. We pointed out that it is then only consistent to also assume in the same spirit that (ii) ownership of a $\$$-stock is more favorable for a €-based investor than for a $\$$-based investor. In other words, *"the €-guy wants the $\$$-stock more than the $\$$-guy"*. While assumption (i) is important in order to explain market prices of cross currency swaps, it has been pointed out that

assumption (ii) is important in order to explain the market prices of equity forwards.

Appendix: Derivation of Formula (2)

For the sake of a simplified notation, but without loss of generality, we assume flat interest rates throughout the proof, i.e. $r_C(t) \equiv r_C$ and $r_x^y(t) \equiv r_x^y$ for all $x, y \in \{\text{€}, \$\}$. We first assume that collateral is posted in $\$$. We denote the value of the derivative at time $t \in [0, T]$ by V_t. This value has two components, namely the final payoff and the PnL on the collateral account, whose value at t is denoted by C_t. The case $C_t > 0$ is interpreted as having received collateral from our counterparty, and $C_t < 0$ as having posted collateral to our counterparty. The assumption of a continuous, zero-threshold two-way CSA simply means that $C_t = V_t$ at all times $t \in [0, T]$. In particular – and this is a decisive difference to the uncollateralized case – the $\$$-value $C_t = V_t$ must be agreed upon by both counterparties at each point in time t, i.e. the law of one price must hold. The rehypothecation assumption implies that the collateral account can be invested (by the €-based investor) at the risk-free rate $r_{\text{€}}^{\$}$. With these notations, the value of the collateralized derivative (for the €-based investor) is given by

$$V_t = \underbrace{e^{-r_{\text{€}}^{\$}(T-t)} \mathbb{E}_t^{\mathbb{Q}}[f(S_T)]}_{\text{payoff part}} + \underbrace{\mathbb{E}_t^{\mathbb{Q}}\Big[\int_t^T C_u\,(r_{\text{€}}^{\$} - r_C)\,e^{-r_{\text{€}}^{\$}(u-t)}\,\mathrm{d}u\Big]}_{\text{collateral part}}$$

$$= e^{-r_{\text{€}}^{\$}(T-t)} \mathbb{E}_t^{\mathbb{Q}}[f(S_T)] + \mathbb{E}_t^{\mathbb{Q}}\Big[\int_t^T V_u\,(r_{\text{€}}^{\$} - r_C)\,e^{-r_{\text{€}}^{\$}(u-t)}\,\mathrm{d}u\Big].$$

This manifests an equation for the value V_t. The solution is given by

$$V_t = e^{-r_C(T-t)} \mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big],$$

as can be checked by plugging it into the right-hand-side of the equation, applying Fubini's Theorem and simplifying the terms. The very same logic can be applied to derive V_t from the point of view of the $\$$-investor, yielding the identical result (replacing simply $r_{\text{€}}^{\$}$ by $r_{\$}^{\$}$ in all computations).

Under the assumption that collateral is posted in € (even though the contract is denoted in $\$$), we must briefly reflect on the collateral amount that both parties agree on. They should agree on the fact that the $\$$-equivalent of the €-value C_t at all times t equals the $\$$-value V_t of the contract, i.e. $C_t\,FX_t = V_t$, where FX_t denotes the value of one € in $\$$ at time t. Consequently, when the €-based investor evaluates the collateral part of the derivative, he may decide between discounting its expected €-value at the rate $r_{\text{€}}^{\text{€}}$ or discounting its expected $\$$-value at the rate $r_{\text{€}}^{\$}$. Since the latter choice does not involve the variable FX_t, it is a little more convenient, and the pricing equation becomes

$$V_t = e^{-r_{\text{€}}^{\$}(T-t)} \mathbb{E}_t^{\mathbb{Q}}[f(S_T)] + \mathbb{E}_t^{\mathbb{Q}}\Big[\int_t^T V_u\,(r_{\text{€}}^{\text{€}} - r_C)\,e^{-r_{\text{€}}^{\$}(u-t)}\,\mathrm{d}u\Big].$$

Notice that the sole difference to the $\$$-collateral case is the rate $r_{\text{€}}^{\text{€}}$ that can be earned on the collateral. This changes the solution

to

$$V_t = e^{-(r^\$_\euro - r^\euro_\euro + r_C)\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big],$$

as can be checked. In particular, the value for the \$-based investor is again the same. For him, the analogous derivation leads to the value

$$V_t = e^{-(r^\$_\$ - r^\euro_\$ + r_C)\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big],$$

but the rates $r^\$_\$ - r^\euro_\$$ and $r^\$_\euro - r^\euro_\euro$ are identical (namely equal to $r^\$_\$ - r^\euro_\euro - x$).

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