TRANCHE ROUND TRIP: DEPENDENCE MATTERS!

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Abstract When CDS insurance on disjoint loss tranches of an index of credit-risky assets is offered, insurance on the whole basket can be aquired by entering into one insurance contract for each and every tranche. Alternatively, one may buy such insurance more directly in terms of entering into a so-called portfolio index CDS contract. In economic terms the two investments appear to be equivalent (except for small technical differences), and hence should come at a similar price. If this is not the case, one may sell the more expensive of the two investments and buy the cheaper in order to log in a seemingly riskfree gain. This investment strategy is known as "tranche round trip" in the marketplace. However, we point out that if the standardized running coupons of the tranche contracts differ, the two investment strategies might be far from being equivalent. In particular, the former depends both on the idiosyncratic default probabilities of the credit-risky assets and the dependence structure between the assets, while the latter index CDS contract is completely unaffected by the dependence structure.

1 The mechanics of a CDO

A collateralized debt obligation (CDO) is an insurance contract between two parties, referring to an underlying pool of credit-risky assets. The insurance seller receives periodic premium payments by the insurance buyer. In return, she is commited to make default compensation payments in case some of the assets in the reference basket default. We denote by $X_1, \ldots, X_d$ the random future time points when these credit events take place, where d denotes the number of underlying assets. If asset k has portfolio weight ω_k and all asstes are assumed to have the identical and constant[1] recovery rate $R \in [0, 1]$, the relative portfolio loss until time $t > 0$ is given by

$$(1-R)\, L_t := (1-R) \sum_{k=1}^{d} \omega_k\, 1_{\{X_k \leq t\}}.$$

The weights ω_k are constant and sum up to one, i.e. $\sum_{k=1}^{d} \omega_k = 1$. In standardized baskets all assets are equally weighted, e.g. $d = 125$ and $\omega_1 = \ldots = \omega_{125} = 0.008$ in case of the Markit

[1] In reality, the recovery rates R_k, for all assets $k = 1, \ldots, d$, are random variables just like the default times X_k. However, for the reason of simplified notation we make this assumption, which is quite common in the marketplace.

iTraxx index. The expected value of the relative portfolio loss is given by

$$\mathbb{E}[(1-R)\,L_t] = (1-R)\sum_{k=1}^{d} \omega_k\,\mathbb{P}(X_k \leq t). \tag{1}$$

Regarding mathematics, the probability distribution of the involved random vector $(X_1, \ldots, X_d)$ allows for a decomposition into the d marginal default probabilities $t \mapsto \mathbb{P}(X_k \leq t)$, $k = 1, \ldots, d$, and a so-called *copula function* $C : [0,1]^d \to [0,1]$, which encodes all information about the stochastic dependence between the default times. This mathematical result is called *Sklar's Theorem*, due to Sklar (1959). As we observe from (1), for the computation of the expected value of the relative portfolio loss no knowledge about the copula of the default times is required at all. This is an important observation because the pricing of a so-called portfolio index CDS contract relies solely on the computation of such expected values, and hence is independent of the interrelation between the assets.

1.1 Portfolio index CDS

Ignoring accrued interest upon default, assuming unit notional, and assuming premium payments at time points $t_1 < t_2 < \ldots < t_n$ with contract maturity t_n, and with the convention $t_0 := 0$, the value of the expected discounted payments of the insurance buyer in a portfolio index CDS contract – the so-called *premium leg (PL)* – is given by[2]

$$PL = u + s\sum_{j=1}^{n}(t_j - t_{j-1})\,DF(t_j)\,(1 - \mathbb{E}[L_{t_j}]),$$

where s is the so-called *index CDS spread*, i.e. the insurance premium per unit notional, and u is an upfront payment (per unit notional) to be made at contract settlement. In return, the expected discounted value of the compensation payments the insurance seller has to make to the protection buyer – the so-called *default leg (DL)* or *protection leg* – is[3]:

$$DL = (1-R)\sum_{j=1}^{n} DF(t_j)\,(\mathbb{E}[L_{t_j}] - \mathbb{E}[L_{t_{j-1}}]).$$

Concerning quotation in the marketplace, the spread s is typically standardized, e.g. $s = 100$ bps, and the upfront payment to be made at inception is then determined so that $PL = DL$, yielding

$$u = \sum_{j=1}^{n} DF(t_j)\,\big((1-R)\,(\mathbb{E}[L_{t_j}] - \mathbb{E}[L_{t_{j-1}}]) \\ - s\,(t_j - t_{j-1})\,(1 - \mathbb{E}[L_{t_j}])\big).$$

1.2 CDO tranches

A CDO tranche contract is similar to a portfolio index CDS contract, the difference being that the insurance buyer receives default compensation payments only in case the relative portfolio loss

[2] We denote by $DF(t)$ a discount factor for the future time point $t > 0$.

[3] Again ignoring accrued interest upon default and assuming default payments only at the end of each quarter in order to simplify our presentation.

$(1-R)\,L_t$ exceeds a contractually specified *attachment point* x, and furthermore does not receive compensation payments for losses exceeding a *detachment point* y, with $0 \leq x < y \leq 1$. Such a contract appears to be a generalization of a portfolio index CDS in the sense that the latter is included by setting $x = 0$ and $y = 1$. However, there is a small technical difference, because the remaining nominal in a tranche contract takes the recovery rate into account while this is not the case for the index CDS[4]. For the pricing of such a contract, relevant is not the overall loss L_t of the portfolio, but only the loss of the considered tranche $[x, y]$, given by

$$L_t^{x,y} := \min\{\max\{0, (1-R)\,L_{t_j} - x\}, y - x\}. \tag{2}$$

When x, y are not zero and one, respectively, the mathematical treatment becomes more involved, because the pricing of a tranche contract then does depend on the copula C of the default times, i.e. a dependence model is required. This follows from the fact that unlike the expected value $\mathbb{E}[(1-R)\,L_t] = \mathbb{E}[L_t^{0,1}]$, the now relevant expected value $\mathbb{E}[L_t^{x,y}]$ for arbitrary x, y cannot be computed as easily as was the case in (1), where we simply used the linearity of the expectation. Now, the peculiar min-max-function in (2) is non-linear, implying that not only the marginal laws of the default times matter, but also their copula. The (again simplified) premium and default legs of protection buyer and seller are now given by

$$\begin{aligned} PL_{x,y} &= u_{x,y}\,(y-x) \\ &\qquad + s_{x,y} \sum_{j=1}^{n} (t_j - t_{j-1})\,DF(t_j)\,(y - x - \mathbb{E}[L_{t_j}^{x,y}]), \\ DL_{x,y} &= \sum_{j=1}^{n} DF(t_j)\,(\mathbb{E}[L_{t_j}^{x,y}] - \mathbb{E}[L_{t_{j-1}}^{x,y}]), \end{aligned}$$

where $u_{x,y}$ and $s_{x,y}$ are the respective tranche upfront and tranche spread, with the analogous meaning as in the index CDS case. In the marketplace, the overall loss interval $[0, 1]$ is typically partitioned into disjoint subintervals, i.e. by introducing tranche attachment/detachment points $0 =: x_0 < x_1 < \ldots < x_{m-1} < x_m := 1$. An investor is then offered the tranches $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{m-1}, x_m]$. In particular, the investor may buy protection for every tranche, i.e. enter into m contracts at a time, a strategy we call *synthetic (portfolio) index CDS* in the sequel. This provides him with default protection for the whole basket, like the portfolio index CDS does. Indeed, we have

$$(1-R)\,L_t = \sum_{\ell=1}^{m} L_t^{x_{\ell-1},x_\ell}, \text{ and hence } DL = \sum_{\ell=1}^{m} DL_{x_{\ell-1},x_\ell},$$

implying that the protection legs of index CDS and synthetic index CDS coincide, since even the payment streams coincide. Regarding the premium legs of the index CDS and the tranche contracts, however, the payment streams do not coincide in general.

[4] Below in Formula (2), the recovery rate R enters the definition of $L_t^{x,y}$, while it does not enter the definition of L_t. We have $L_t^{0,1} = (1-R)\,L_t \neq L_t$.

We observe that

$$
\begin{aligned}
PL = DL = \sum_{\ell=1}^{m} DL_{x_{\ell-1},x_\ell} &= \sum_{\ell=1}^{m} PL_{x_{\ell-1},x_\ell} \\
&= \sum_{\ell=1}^{m} u_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1}) \\
&\quad + \sum_{j=1}^{n}(t_j - t_{j-1}) \sum_{\ell=1}^{m} s_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1} - \mathbb{E}[L_{t_j}^{x_{\ell-1},x_\ell}]).
\end{aligned} \tag{3}
$$

If all standardized tranche spreads were equal, i.e. if $s_{x_0,x_1} = \ldots = s_{x_{m-1},x_m} =: s_0$, this would imply the equation

$$
\begin{aligned}
u - \Big(\sum_{\ell=1}^{m} u_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1})\Big) \\
= (s_0 - s)\,\sum_{j=1}^{n}(t_j - t_{j-1})\,DF(t_j)\,(1 - \mathbb{E}[L_{t_j}]) \\
+ s_0\,R\,\sum_{j=1}^{n}(t_j - t_{j-1})\,DF(t_j)\,\mathbb{E}[L_{t_j}],
\end{aligned}
$$

which depends only on the expectation values of L_t. Hence, without the use of a copula model it can be checked whether the given spread and upfront quotes for tranches and index are fair. However, if the standardized tranche spreads are different, (3) constitutes a non-trivial equation. In particular – and this is what we like to point out in the present note – the premium leg of the synthetic index CDS depends on the copula of the default times, while the premium leg of the proper index CDS does not! Consequently, Equation (3) adds a non-negligible consistency condition that must be met when choosing a particular copula for the model to be used. Conversely, if one already has decided for a specific copula model (e.g. by fitting it to all tranche quotes), plugs it into the right hand side of Equation (3), and then observes a violation of Equation (3), it might be a lucrative investment to go long the synthetic index CDS and short the proper index CDS, or vice versa, depending on which premium leg is cheaper – a strategy called *tranche round trip* in the marketplace. Note, however, that this is not a strict arbitrage, because the payment streams of synthetic and proper index CDS, i.e. the initial upfronts and the running coupon payments to be made, differ. An explicit example of the payment streams is provided in the upcoming section.

2 A motivating example

Consider the following contracts offered in the marketplace in the end of October 2013 on a certain series of the Markit iTraxx index with five year maturity:
With $x_0 := 0$, $x_1 := 3\%$, $x_2 := 6\%$, , $x_3 := 9\%$, $x_4 := 12\%$, $x_5 := 22\%$, $x_6 := 100\%$, from Table 1 we readily compute the upfront payment of the synthetic index CDS to equal

$$\sum_{\ell=1}^{6} u_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1}) \approx 1.2324\%,$$

which is higher than the upfront of the proper index CDS. Accordingly, we expect the “average” spread level of the synthetic

product	spread in bps	upfront in %
index CDS	100	-0.8607
tranche $[0\%, 3\%]$	500	21.0000
tranche $[3\%, 6\%]$	500	-4.6250
tranche $[6\%, 9\%]$	100	6.3898
tranche $[9\%, 12\%]$	100	2.5021
tranche $[12\%, 22\%]$	25	2.5908
tranche $[22\%, 100\%]$	25	0.2760

Table 1: Market ask quotes in the end of October 2013 for a certain series of the Markit iTraxx index with five year maturity.

index CDS to be less than 100 bps, which is the spread level of the true index CDS. Indeed, the spread level of the synthetic index CDS is lower, and it depends on the remaining nominal in the portfolio, i.e. changes upon a default event. More clearly, assuming a recovery rate of $R = 40\%$ it is given by

$$\sum_{\ell=1}^{6} s_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1}) = 58 \text{ bps.}$$

until the first default is observed, and then

$$0.05\left(0.03 - \frac{1-R}{125}\right) + \sum_{\ell=2}^{6} s_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1}) = 55.6 \text{ bps.}$$

until the second default is observed, and then

$$0.05\left(0.03 - 2\,\frac{1-R}{125}\right) + \sum_{\ell=2}^{6} s_{x_{\ell-1},x_\ell}\,(x_\ell - x_{\ell-1}) = 53.2 \text{ bps.}$$

until the third default is observed, and so on. Why does the spread level decrease upon a default event? The reason is that the first defaults only affect the lowest tranche $[0\%, 3\%]$, and hence only the spread of this tranche alone has to be paid on a fewer remaining tranche nominal after the default. Since the tranche spread of 500 bps is higher than the average tranche spread (the higher tranches have a smaller spread) this reduction of remaining nominal in the equity tranche results relatively in an overall reduction of the spread to be paid on the whole synthetic index CDS.

Regarding quantitative statements, within the presented example Figure 1 shows the value of the premium leg for the synthetic index CDS for different levels of dependence. More clearly, the marginal laws $t \mapsto \mathbb{P}(X_k \leq t)$ are held fix and dependence is modeled in terms of the copula

$$C_\alpha(u_1, \ldots, u_d) = \prod_{k=1}^{d} u_{[k]}^{k^{(1-\alpha)} - (k-1)^{(1-\alpha)}}, \tag{4}$$

where $u_{[1]} \leq \ldots \leq u_{[d]}$ denotes the ordered list of the arguments $u_1, \ldots, u_d \in [0, 1]$. This is a so-called *Lévy-frailty copula*[5], which

[5] See Mai, Scherer (2009) for details.

includes the boundary cases C_0 of independence and C_1 of full comonotonicity. A comparison of this model with the commonly applied Gaussian market model can be found in Mai (2013) and Bernhart (2013). As one can clearly see in Figure 1, the value of the premium leg of the synthetic index CDS – unlike the premium leg PL of the ordinary index CDS – is not invariant under the choice of the dependence parameter α. Only for $\alpha \approx 0.4282$ we observe that it equals the default leg DL, and, hence, the premium legs of ordinary and synthetic index CDS agree (because $PL = DL$ by definition). However, as can be seen from the (small) range of the y-axis in Figure 1, the dependence parameter has only a weak, though not negligible, effect.

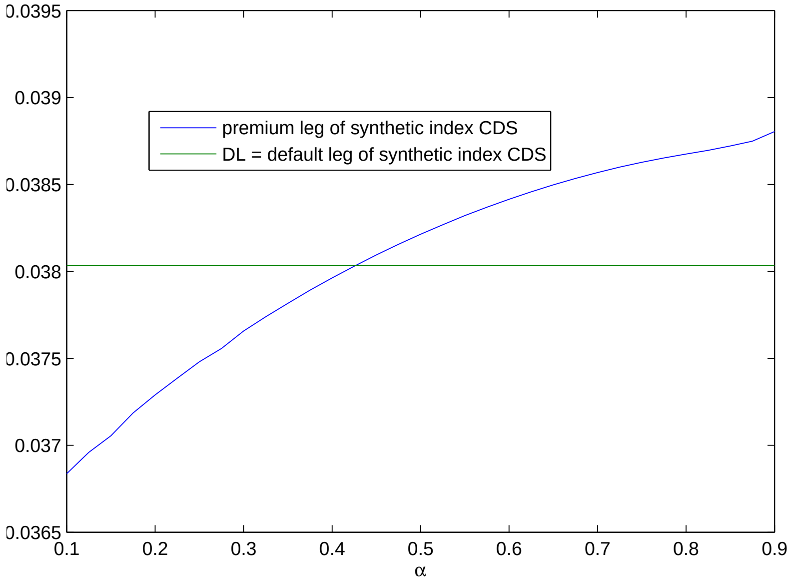

Fig. 1: Dependence of the premium leg for the synthetic index CDS with respect to the level of dependence. Dependence is modeled by means of a one-parametric copula C_α, where α ranges from zero (independence) to one (comonotonicity).

In order to be able to explain all quoted tranche prices within one modeling framework, it has become common practice to use the so-called *base correlation concept*, see McGinty (2004) for a detailed explanation. This concept is directly transferable to the copula model (4) we used in our example above. This yields a “base α curve” instead of a base correlation curve, the latter terminology stemming from a modeling setup in which the copula (4) is replaced with the more commonly known one-factor Gaussian copula. The base α curve consists of the values $\alpha_1, \ldots, \alpha_{m-1}$, which are defined recursively in such a way that when the computation of any principal tranche $[0, x_L]$ is priced with the parameter α_L for $L = 1, \ldots, m-1$, then the observed market tranche prices – which can be priced as differences of two principal tranches – are perfectly explained. Traditionally, the base α curve ends at the detachment point x_{m-1}, because the remaining super senior tranche $[x_{m-1}, 1]$ is intuitively thought of as being perfectly explained by a model that explains the tranche $[0, x_{m-1}]$ and the index CDS jointly. This is because it is the complementary tranche which from an economic point of view is thought to be equivalent to a portfolio consisting of the index CDS minus a contract for tranche $[0, x_{m-1}]$. However, our example above shows that this way of thinking is misleading. In order to circumvent this problem, the base α curve must be extended by one more element referring to the super senior tranche.

3 Conclusion

It was explained how market conventions with respect to different standardized strike levels for CDO tranches trigger the necessity to extend the usual base correlation framework by a further point referring to the super senior tranche.

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