XAIA INVESTMENT

The negative basis conditioned on default

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Abstract The negative basis is an annualized earnings figure that measures the expected excess return of a bond investment over a reference discounting rate, after credit risk has been hedged away via credit default swap (CDS) protection. Unfortunately, the most appropriate measurement of the negative basis is in terms of a root of a non-trivial decreasing function. The goal of the present article is to derive closed proxy formulas for the negative basis, at least in special situations. Under the assumption that a credit event before maturity of bond (and CDS) is certain, i.e. conditioned on default before maturity, we are able to derive a very simple formula. This may help to estimate a negative basis for very long-dated bonds, when default becomes certain under the assumption of perpetual bond and CDS maturity. Furthermore, a small enhancement of the formula provides a proxy formula for the (unconditional) negative basis. The availability of such closed formulas allows to study qualitative properties of the negative basis, such as dependence on the recovery rate assumption or leg prices.

1 Introduction

Typical credit-risk pricing models often cannot jointly explain observed market prices for bonds issued by some company and credit default swaps (CDS) referencing on the same company. This is due to the fact that they only model credit-risk and interest rate risk, but ignore secondary risk factors that affect the prices of bonds and CDS in different ways. The most dominant ignored factor is liquidity, which can put pressure on bond (CDS) prices while not affecting CDS (bond) prices, cf. Longstaff et al. (2005). Measuring observed discrepancies between bond and CDS prices after credit-risk has been taken into account is an interesting topic, in particular because the observed discrepancies can actually be traded. Such trading strategies are called *basis trading*. A negative basis strategy tries to profit from too cheap CDS protection by buying bond and CDS protection. Conversely, a positive basis strategy tries to profit from too expensive CDS protection by selling protection and short-selling the bond. Since the latter is technically more difficult to execute in practice, we focus on negative basis strategies in the sequel. The term "negative basis" informally denotes an annualized return figure that can be earned on the bond investment after having hedged away credit risk completely via CDS protection.

The most common negative basis measurements define it as the difference between some bond spread and the CDS par spread, e.g. Zhu (2004); Blanco et al. (2005); De Wit (2006); Choudhry (2006); Bühler, Trapp (2009); Bai, Collin-Dufresne (2011); Palladini, Portes (2011); Li et. al (2011). Since bond and CDS spreads are derived from the respective bond and CDS prices via quite different procedures, the resulting detection of price discrepancies is blurred masssively, in particular dependent on the definition of the applied bond spread. Furthermore, since negative basis measurement in theory should detect only the price discrepancies modulo the joint credit risk contained in both products, a definition in terms of a difference between two spreads that are calculated for each product separately must necessarily be flawed. Bernhart, Mai (2016) explain why such measurements can only be rudimentary and do not fit well into the theory of Mathematical Finance, and propose a superior method called *hidden yield approach*.

Unfortunately, the hidden yield negative basis measurement must be computed numerically and thus cannot be analyzed easily from a closed formula. The goal of the present article is to provide a simple closed formula for the hidden yield negative basis, at least in a special situation. This goal is achieved under the idealized assumptions of continuous coupon payments and flat discounting curve, when assuming that a default before maturity of bond (and CDS) is certain. Letting the maturity tend to infinity, the formula becomes valid as an (unconditional) hidden yield negative basis measurement for perpetual bond and CDS. Qualitative properties, especially regarding its sensitivity on observed input parameters, can be studied thoroughly from such a closed formula, helping to broaden our understanding of the hidden yield negative basis measurement.

The remainder of the present article is organized as follows. Section 2 introduces the applied notations and assumptions. Section 3 justifies the presented formula for the negative basis conditioned on default from basic profit and loss considerations, based on nothing but pure cashflow computations. Section 4 provides a second justification for the presented formula in that it relates it to a risk-free discounting rate of a credit-risk pricing model. Having justified the formula from two different viewpoints, Section 5 studies properties of the formula and as a major application derives a quick-and-dirty formula for the unconditional hidden yield negative basis. Finally, Section 6 concludes.

2 Notations and assumptions

We consider a financial market with three traded assets: a risk-free bank account, a bond, and a maturity-matched CDS. This financial market is fully specified by seven parameters

$$(u, p, s, c, R, r, T) \in [-1, 1] \times (0, \infty)^3 \times [0, 1] \times \mathbb{R} \times (0, \infty)$$

with the following interpretations:

- u: price (= upfront) of the CDS.
- p: price of the bond.
- s: coupon rate of the CDS.
- c: coupon rate of the bond.
- R: recovery rate of the bond (and the CDS).

- r: risk-free rate, used for discounting cash flows.
- T: maturity of bond and CDS.

In order to be able to derive a closed formula for the negative basis conditioned on default, we need to assume that the bond coupon rate c and the CDS coupon rate s are paid continuously. Clearly, in practice these payments occur periodically, so our assumptions constitute an idealization. However, this approximation allows us to get rid of technicalities such as discrete sums and accrued coupon adjustments that prevent the derivation of closed formulas. Since required in later calculations, we postulate the following bounds (A) on the prices u and p, which are backed by obvious economic reasoning and are virtually always met in practice. More precisely, we assume

$$-s \int_0^T e^{-r\,t}\,\mathrm{d}t < u < 1 - R, \quad R < p. \tag{A}$$

As a main contribution, the main body of the present article justifies the following formula for the negative basis under the assumption that a credit event before T is certain:

$$\boxed{b := c\,\frac{1 - R - u}{p - (p+u)\,R} - s\,\frac{p - R}{p - (p+u)\,R} - r.} \tag{NBD}$$

There is one noticeable observation to be made from the formula: if the basis package trades at par, i.e. if $p + u = 1$, then the negative basis conditioned on default boils down to $c - s - r$. Hence, in this case one only needs to compare bond coupon with CDS coupon, and benchmark their difference against a reference discounting rate. If the package trades away from par, then the negative basis conditioned on default changes from this simple formula $c - s - r$ to $w_1\,c - w_2\,s - r$, where the weights w_1, w_2 depend on the recovery rate as well as bond and CDS prices. Consequently, we may deduce heuristically from Formula (NBD) that negative basis measurement is composed of two (separate) tasks: firstly by weighting bond coupon and CDS coupon appropriately, according to prices and recovery assumptions, and secondly by choosing an appropriate benchmark rate r. While the appropriate choice of a benchmark rate r is abstracted from in the present study via the assumption of a flat discounting curve, see also the following remark, the closed formula for the coupon weights is a theoretical result that might provide valuable insights for negative basis measurement in general.

Remark 2.1 (On the assumptions made)

- **Recovery rate:** One might be bothered by the assumption that bond recovery and CDS recovery are identical. However, if a negative basis investor delivers her bonds into the auction following a CDS credit event, this is precisely the outcome. In other words, a negative basis investor can force bond and CDS recovery to be identical for her purpose. When doing so, she only dismisses a potential recovery upside in case the bond recovers higher than the CDS. But in order to realize this upside she must bet on a favorable recovery mismatch in the

CDS auction, which is a (recovery mismatch) trading strategy that can be considered seperately to the pure negative basis strategy - and which is not considered in the present article. We refer the interested reader to Gupta, Sundaram (2015) for background on recovery mismatch trading.

- **Flat discounting curve:** The rate r serves as a reference rate, relative to which the negative basis is measured. By definition, the negative basis is defined as the spread on top of r that can be earned without exposure to default risk. Keeping this in mind, obviously different choices of r imply different negative bases, and r has to be chosen deliberately. Depending on her funding opportunities, an investor typically discounts a future cashflow at time t at a rate $r(0,t)$, and the zero rate curve $t \mapsto r(0,t)$ is bootstrapped from observed market prices of interest rate related products. In order to derive a closed formula, the present article requires the assumption that $r(0,t) \equiv r$ is a flat curve. As a practical application of the results derived herein we recommend to consider the derived formulas as approximations using the rate $r = r(0,T)$.

- **Coupon payment frequency:** The idealized assumption of continuous coupon payments appears to be innocent, and in fact is not severe in many cases, but it can be quite rough in special situations. In order to demonstrate this, we like to highlight that the hidden yield negative basis measurement in Bernhart, Mai (2016) can depend critically on the payment frequency. Consequently, the approximation of the hidden yield negativ basis of Bernhart, Mai (2016) by the formulas derived in the present article, e.g. (NBD) or (3), are rough in such a situation. Let n denote the number of coupon payments per year (e.g. $n = 1$ in case of annual payments, $n = 2$ in case of semi-annual payments, etc.). In order to protect against losses resulting from a credit event immediately before the maturity T, it is necessary to buy at least $1 + c/n$ times the bond nominal in CDS nominal with maturity greater or equal to T. The case of continuous coupon payments corresponds to $n \to \infty$, so that a nominal-matched CDS suffices. However, in practice $n < \infty$ and it might occur that the additional CDS cost $c/n \cdot u$ has non-negligible size. In such a case the required initial cost for the optimal CDS-hedge is significantly higher in case of discrete coupon payments than in case of continuous coupon payments. To provide a numeric example that is motivated by a real-world case, consider the parameters

$$(p, u, c, s, R, T, r) = (0.31, 0.68, 0.06625, 0.05, 0.1, 5, -0.001).$$

  While Formula (NBD) yields a value of 203 bps, the hidden yield method proposed by Bernhart, Mai (2016) yields a negative basis of 180 bps in case of semi-annual bond coupon payments for these parameters, and of 120 bps in case of annual bond coupon payments (CDS coupons quarterly in both cases). This indicates how strong the effect of payment frequency can be in special situations with high CDS upfront costs.

3 Profit and loss considerations

Throughout, we denote by τ the random future time point of a CDS credit event, at which the bond is delivered into the CDS auction, which is assumed to spit out the recovery rate R. A portfolio consisting of one unit bond nominal and $\alpha > 0$ units CDS nominal - and holding it until $\min\{\tau, T\}$ - generates the following PnL, discounted back into today:

$$\text{PnL}(\tau) = -p - \alpha\, u + \begin{cases} \int_0^{\tau} (c - \alpha\, s)\, e^{-r\,t}\, \mathrm{d}t + \big(\alpha + R\,(1-\alpha)\big)\, e^{-r\,\tau} & \text{, if } \tau \leq T \\ \int_0^{T} (c - \alpha\, s)\, e^{-r\,t}\, \mathrm{d}t + e^{-r\,T} & \text{, if } \tau > T \end{cases}.$$

Now PnL(τ) is a random variable depending on the unknown future time point of a credit event. If a credit event happens immediately, i.e. if $\tau = 0$, the value

$$\text{PnL}(0) = -p - \alpha\, u + \alpha + R\,(1-\alpha)$$

is called the *Jump-to-Default* of the portfolio. Postulating a default-risk free portfolio and setting the last equation equal to zero, the Jump-to-Default neutral CDS-hedge ratio is found to be

$$\alpha = \frac{p - R}{1 - u - R},$$

which is well-defined and positive since $p - R > 0$ and $1 - u - R > 0$ by Assumption (A). Using this particular hedge ratio α in the sequel, the initial investment cost equals

$$p + \alpha\, u = \frac{p - (p+u)\,R}{1 - R - u}.$$

This initial investment cost is positive: if $u \geq 0$, then this follows immediately from $\alpha > 0$ and $p > R \geq 0$ by considering the left-hand side of the last equation. If $u < 0$, then this follows from observation of the numerator on the right-hand side of the last equation, which satisfies $p - (p+u)\,R > (1-R)\,p > 0$. The profit and loss PnL(τ), relative to the initially invested capital, can be rewritten as

$$\frac{\text{PnL}(\tau)}{p + \alpha\, u} = \begin{cases} b\, \int_0^{\tau} e^{-r\,t}\, \mathrm{d}t & \text{, if } \tau \leq T \\ b\, \int_0^{T} e^{-r\,t}\, \mathrm{d}t + \frac{(1-R)\,(1-(p+u))}{p-(p+u)\,R}\, e^{-r\,T} & \text{, if } \tau > T \end{cases}. \tag{1}$$

In particular, it is observed that PnL(τ) is almost surely non-negative if and only if $b > 0$ (from the case $\tau \leq T$) and (from the survival case $\tau > T$)

$$p + u \leq \frac{p\, b\, \int_0^{T} e^{-r\,t}\, \mathrm{d}t + e^{-r\,T}\,(1-R)}{R\, b\, \int_0^{T} e^{-r\,t}\, \mathrm{d}t + e^{-r\,T}\,(1-R)} =: B. \tag{2}$$

Formula (1) provides a first justification to call the rate b a *negative basis conditioned on default*, because it can be earned until τ on one's initially invested capital, provided τ happens before T. A second justification, relating it to the hidden yield methodology of Bernhart, Mai (2016) is provided in the subsequent section. The additional term in the survival case $\tau > T$ in (1) corresponds to an adjustment that is due to the pull-to-par effect of the basis package. To this end, it is illuminating to notice that the CDS-hedge ratio α is greater than one if and only if the package trades above par, i.e. $\alpha > 1 \Leftrightarrow p + u > 1$. If the package trades even above the upper bound $B > 1$ in (2), then it is impossible to have an almost surely positive PnL, even though $b \geq 0$. This is because in case of survival until T the cost for the instantaneous Jump-to-Default neutral CDS hedge with the ratio $\alpha > 1$ implies running costs that ultimately make the PnL negative. The basis package is expected to pull to par until maturity T, and in practice one should reduce α every now and then according to the already realized pull-to-par in order to minimize these costs.

4 Pricing model considerations

So far, the argumentation was completely model-free in the sense that there was no need to consider a credit-risk pricing model, i.e. a model for the random variable τ. Also, the claimed formula for b in (NBD) has not yet been put in relation to the hidden yield negative basis of Bernhart, Mai (2016). In order to achieve this, it is educational to investigate the relation of Definition (NBD) to arbitrage pricing theory, i.e. to study which role b plays within a specific stochastic model for τ.

Arbitrage pricing theory suggests to evaluate the prices for bond and CDS as the expected, discounted sum over all future cashflows to be received from the respective product. When only modeling credit risk stochastically, these expected, discounted cashflows depend both on the stochastic model for τ and the chosen discounting rate $r + x$, which we providently denote as the sum of the reference rate r and an a priori unknown spread x. In mathematical terms, the model prices for bond and CDS are given by

$$p_T(x,\tau) = \mathbb{E}\Big[c \int_0^{\min\{\tau,T\}} e^{-(r+x)\,t}\,\mathrm{d}t + e^{-(r+x)\,T}\,1_{\{\tau>T\}} + R\,e^{-(r+x)\,\tau}\,1_{\{\tau\leq T\}}\Big],$$

$$u_T(x,\tau) = \mathbb{E}\Big[(1-R)\,e^{-(r+x)\,\tau}\,1_{\{\tau\leq T\}} - s\int_0^{\min\{\tau,T\}} e^{-(r+x)\,t}\,\mathrm{d}t\Big].$$

In the following lemma, we assume that τ has an exponential distribution with parameter $\lambda > 0$. In this case, we write $p_T(x,\tau) = p_T(x,\lambda)$, respectively $u_T(x,\tau) = u_T(x,\lambda)$. In order to proceed, we denote a potential root of the strictly decreasing function

$$f(x) := u + s\int_0^T e^{-(r+x)\,t}\,\mathrm{d}t, \quad x \in \mathbb{R},$$

by $b_*(T)$. If $u \geq 0$, then f has no root and we conveniently set $b_*(T) = \infty$. If $u < 0$, then f has a positive root $b_*(T) > 0$ by the lower bound on u in Assumption (A). The proof of the following lemma requires the following technical assumption:

$$-\frac{c\,u}{s} + e^{-\big(r+b_*(T)\big)\,T} < p. \tag{TA}$$

Lemma 4.1 (Relation of b to credit risk pricing model)
Assume that (TA) holds. There is a unique pair of real numbers $(b_T,\lambda_T) \in \mathbb{R}\times(0,\infty)$ satisfying

$$p_T(b_T,\lambda_T) = p, \quad u_T(b_T,\lambda_T) = u.$$

This statement also holds for all maturities $\tilde{T} \geq T$, and with $\lambda := (c - p\,(r+b))/(p-R) > 0$ we have

$$\lim_{\tilde{T}\to\infty} (b_{\tilde{T}}, \lambda_{\tilde{T}}) = (b, \lambda).$$

Proof
See the Appendix. □

Remark 4.2 (On Assumption (TA))
In most practical cases we have that $b_*(T) = \infty$, so that Condition (TA) boils down to the assumption $c\,u + s\,p > 0$. The latter condition is satisfied in all practical cases of interest. It can be viewed as a slight strengthening of the totally reasonable lower bound on u in (A), since the estimate $p \leq \int_0^T c\,e^{-r\,t}\,\mathrm{d}t$ is met in almost all cases of practical interest. The proof of Lemma 4.1 shows that $c\,u + s\,p > 0$ already guarantees positivity of λ but the existence of a unique solution (b_T, λ_T) with positive λ_T for $T < \infty$ may not be guaranteed. For the sake of completeness, we provide an example of parameters violating the condition, obviously quite pathological. If we choose $(u, p, c, s, R, r, T) = (-0.04, 0.41, 0.2, 0.01, 0.4, 0, 5)$, then $c\,u + s\,p = -0.0039 < 0$. In this case the negative basis conditioned on default b would have a value of 4882 basis points, but $\lambda < 0$, so it is not related to an exponential random variable.

Remark 4.3 (What is the interpretation of Lemma 4.1?)
For finite maturity T, the number b_T in Lemma 4.1 might be considered a reasonable definition for a negative basis, because it is defined analogously to the hidden yield methodology in Bernhart, Mai (2016), who provide a satisfying justification of their definition. This justification is briefly explained heuristically for the convenience of the reader. Arbitrage pricing theory suggests that the three traded assets of the model (bond, CDS, and risk-free bank account) are arbitrage consistent - provided model prices match market prices, as in Lemma 4.1. In such a pricing model, the market's expectation for the expected annualized return on investment for each of the three assets, or any linear combination of them, enters as a discounting rate related to the risk-free bank account, denoted by $r + x$. Now the risk-free bank account with rate $r+x$ does not exist in reality but only in the model. However, it has been demonstrated in Section 3 that an investment only in bond and CDS (without risk-free bank account) replicates a (credit-)risk-free bank account pretty close. Reversing the logic of arbitrage pricing theory, it is approximately possible to earn the rate $r + x$ by replicating the ficticiously modeled risk-free bank account via the actually tradable assets, namely bond and CDS. Lemma 4.1 now tells us the rate that can be earned by this approach, namely precisely $r + b_T$, i.e. the hidden yield negative basis $x = b_T$ is defined as a spread on the reference discounting rate r that can be earned without (credit-)risk exposure.

5 Applications and exploratory analyses

One of the striking advantages of a closed formula for a quantity of interest is the ability to study its sensitivity with respect to the input parameters. E.g. we can observe that the negative basis conditioned on default is non-decreasing (non-increasing) in the recovery rate assumption for packages trading below (above) par. Moreover, for bonds with very long maturity T the derived formula becomes a good proxy for the unconditional hidden yield basis, since default before T becomes very likely in that case. Consequently, the formula is useful for estimating a hidden yield negative basis for long-dated bonds. Finally, as a major application, we first derive a quick-and-dirty approximation to the unconditional hidden yield negative basis by enhancing the formula for b slightly.

5.1 Proxy for unconditional hidden yield negative basis

From Formula (1), we may deduce a reasonable proxy formula for an unconditional rate of return under the assumption that the hedge ratio α is kept constant until T, namely

$$\boxed{b\,(1-w_T)+\Big(b+\frac{(1-R)\,(1-(p+u))}{p-(p+u)\,R}\;\frac{e^{-r\,T}}{\int_0^T e^{-r\,t}\,\mathrm{d}t}\Big)\,w_T,} \tag{3}$$

where an involved probability weight $w_T := \mathbb{P}(\tau > T) \in [0,1]$ must be chosen reasonably. This formula, e.g. with w_T bootstrapped from observed CDS quotes or even rougher from a trader's gut feeling, can serve as quite a useful quick-and-dirty proxy for the hidden yield negative basis in real-world examples.
Figure 1 shows the values (b_T, λ_T) of Lemma 4.1 in dependence on the maturity T, for an exemplary set of model parameters. The convergence result of Lemma 4.1 for large maturities can be observed pretty well. It is also observed that the maturity T does have a non-negligible impact on b_T. Furthermore, the quick-and-dirty formula (3) to compute a negative basis under the assumption of keeping the Jump-to-Default neutral hedge ratio constant until T is quite close to the value b_T. It has been computed with survival probability $w_T := \exp(-\lambda_T \cdot T)$. In accordance with Formula (1), the value b_T and the quick-and-dirty proxy are both smaller than b, since the package trades above par. Although the rate b is earned instantaneously using the Jump-to-Default-neutral hedge ratio, a part of these earnings are eaten up in case of survival until maturity due to the pull-to-par effect of the package.

5.2 Sensitivity on the recovery rate R

The negative basis conditioned on default b as defined in (NBD) is a bounded and monotone function in R. It is monotonically non-decreasing if and only if

$$\big(1-(u+p)\big)\,(c\,u+s\,p)\geq 0.$$

As already noted in Remark 4.2, $c\,u + s\,p > 0$ holds in all practical cases of interest. Consequently, whether the negative basis conditioned on default is decreasing or increasing in practice depends on whether the package price $u+p$ trades above or below par. In particular, if the package trades at par, then the negative basis conditioned on default equals precisely $b = c - s - r$, irrespectively of the recovery rate R.
In any case, b remains within the two values one obtains when considering the two boundary cases $R = 0$ and $R \nearrow \min\{1, p\}$,

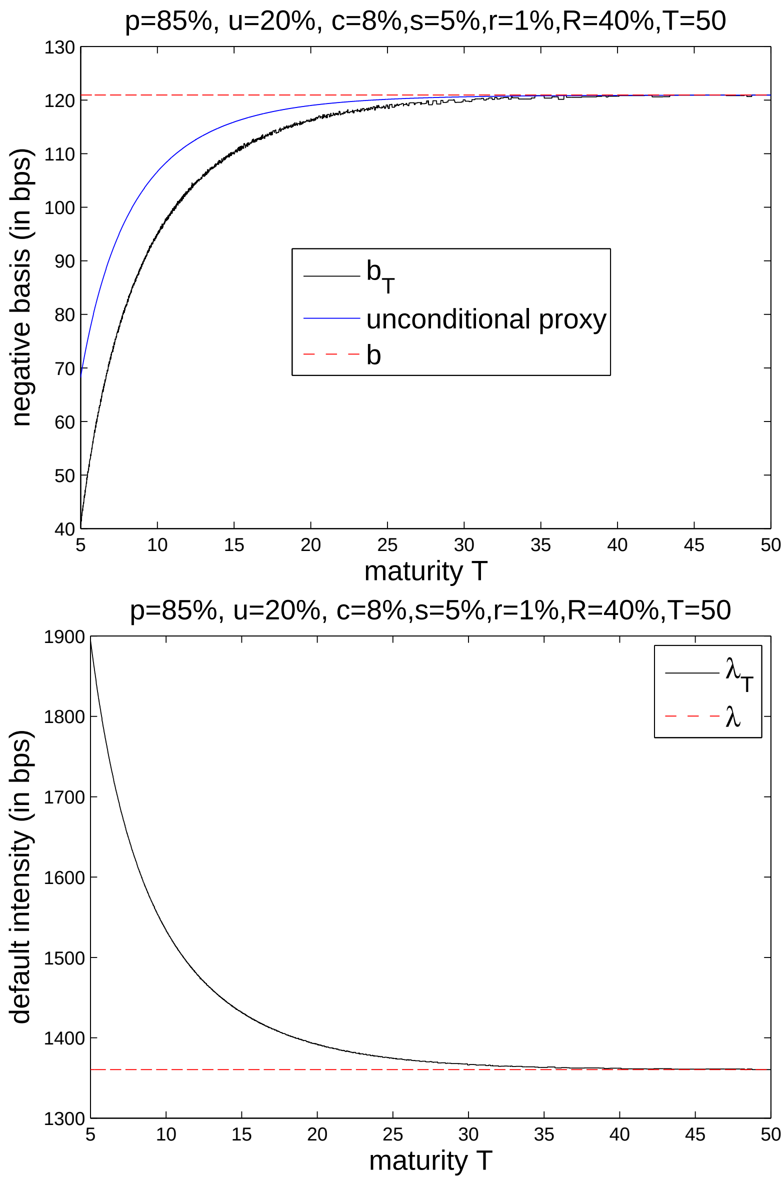

Fig. 1: Top: The values b_T and the quick-and-dirty formula (3) in dependence on the maturity T. Bottom: The value λ_T in dependence on the maturity T.

which are given by

$$\lim_{R\searrow 0} b = c\,\frac{1-u}{p} - s - r,$$

$$\lim_{R\nearrow \min\{1,p\}} b = \begin{cases} -\infty & \text{, if } p>1,\ u=0,\ (u+p\neq 1) \\ c+s\,\frac{p-1}{u}-r & \text{, if } p>1,\ u\neq 0,\ u+p\neq 1 \\ \frac{c}{p}-r & \text{, if } p\leq 1,\ u+p\neq 1 \\ c-s-r & \text{, if } u+p=1 \end{cases}.$$

These limits can be applied together with the following limit relations, in order to yield boundaries for the unconditional hidden yield negative basis approximation via Formula (3):

$$\lim_{R\searrow 0}\frac{(1-R)\,(1-(p+u))}{p-(p+u)\,R}=\frac{1-(p+u)}{p},$$

$$\lim_{R\nearrow\min\{1,p\}}\frac{(1-R)\,(1-(p+u))}{p-(p+u)\,R}=\begin{cases}\frac{1-p}{p} & \text{, if } p>1,\ u=0,\ (u+p\neq 1)\\ 0 & \text{, if } p>1,\ u\neq 0,\ u+p\neq 1\\ \frac{1-p}{p} & \text{, if } p\leq 1,\ u+p\neq 1\\ 0 & \text{, if } u+p=1\end{cases}.$$

5.3 b as a function of u and p

Before an investor actually considers a negative basis investment, obviously she wants to make sure whether the negative basis is positive at all. Intuitively, this is satisfied whenever the package price $u+p$ is not too large. Interestingly, the defining formula (NBD) for the negative basis conditioned on default b provides a linear inequality in the observed prices u and p, namely

$$b>0\ \Leftrightarrow\ u<\frac{c\,(1-R)+s\,R}{c-r\,R}-p\,\frac{s+r\,(1-R)}{c-r\,R},\ c>r\,R.$$

This area is visualized in Figure 2 for a set of sample parameters.

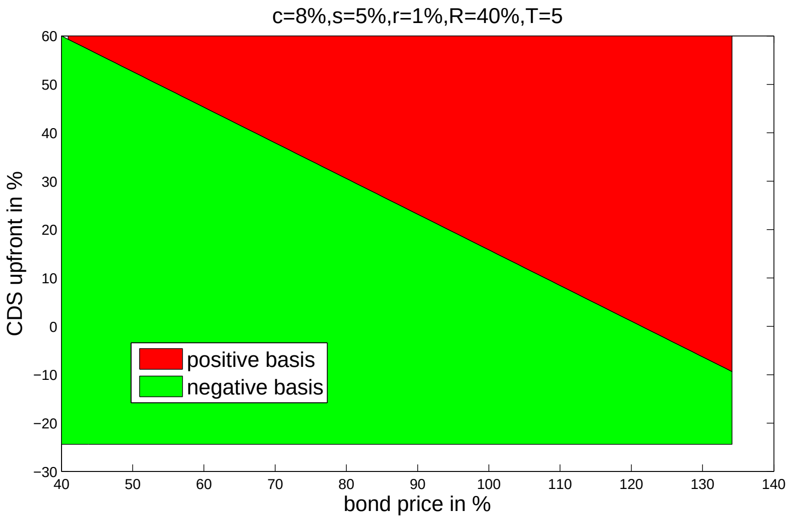

Fig. 2: Visualization of the price pairs (u,p) for which there is negative ($b>0$) and positive ($b<0$) basis conditioned on default.

5.4 Negative basis for long-dated bonds

In practice, CDS protection is typically available only for maturities T smaller or equal to ten years. But there exist CDS-eligible bonds with much longer maturity like 30 or 40 years. In this case, a maturity-matched negative basis investment is impossible, but a maturity-mismatched negative basis investment is possible. This means that one buys the bond and hedges with shorter-dated CDS. Unlike a maturity-matched negative basis investment, such a position can never be free of credit risk. The CDS can at best reduce the bond credit risk, but never eliminate it. Still, a considerable part of the position's expected return potential might be due to a negative basis component, i.e. in addition to the credit risk exposure there might be a liquidity-driven extra return that one wishes to measure. In order to do so, required is an extrapolation assumption on the CDS credit curve to the bond maturity T. Having made such an assumption, the extrapolated upfront price u may be used together with the bond's price p in order to measure the negative basis according to Formula (NBD) or the quick-and-dirty formula (3). Since T is quite large in this case and since it is not uncommon to assume that $\mathbb{P}(\tau > T)$ tends to zero for $T \to \infty$, the latter two formulas are not much different, so that b is actually quite a good proxy for the unconditional hidden yield negative basis in this case. Consequently, for such maturity-mismatched negative basis investments the derived formula for b can serve as a simple, yet useful tool to help measure its attractiveness.

6 Conclusion

Under idealized assumptions like continuous coupon payments and flat discounting curve, a closed formula for the hidden yield negative basis conditioned on default before bond (and CDS) maturity has been obtained. This formula is useful to study qualitative properties of the negative basis and to derive a simple, closed proxy formula for the unconditional hidden yield negative basis, see (3).

Appendix: Proof of Lemma 4.1

We start with the following two auxiliary lemmata.

Lemma 6.1 (Existence of a certain random variable)
Given $(u, p, c, s, R, r) \in [-1, 1] \times (0, \infty)^3 \times [0, 1] \times \mathbb{R}$, subject to (A), there exists a positive random variable τ satisfying

$$\begin{cases} \mathbb{E}\Big[e^{-(r+b)\,\tau}\Big] = \frac{p\,(r+b)-c}{R\,(r+b)-c} & \text{, if } r+b \neq 0 \\ \mathbb{E}[\tau] = \frac{p-R}{c} & \text{, if } r+b = 0 \end{cases}.$$

Proof
The case $r + b = 0$ is trivial, since $(p - R)/c > 0$ by Assumption (A), so we may choose τ as any positive variable with that given mean. In general, we have that

$$R\,(r+b) - c < 0 \;\Leftrightarrow\; c\,(1-R) + s\,R > 0,$$

which is a true statement by assumption. It follows that

$$\frac{p\,(r+b)-c}{R\,(r+b)-c} \begin{Bmatrix} > \\ < \end{Bmatrix} 1 \;\Leftrightarrow\; r+b \begin{Bmatrix} < \\ > \end{Bmatrix} 0,$$

which makes the statement obvious, as τ may be chosen as a constant, for example. □

Lemma 6.2 (Perpetual model bond and model CDS)
Given $(u, p, c, s, R, r) \in [-1, 1] \times (0, \infty)^3 \times [0, 1] \times \mathbb{R}$, subject to (A), and τ a random variable such as in Lemma 6.1. Then the negative basis (NBD) is the unique solution $x = b$ of the equation system

$$p = p_\infty(x, \tau) := \mathbb{E}\Big[c \int_0^\tau e^{-(r+x)\,t}\,\mathrm{d}t + R\,e^{-(r+x)\,\tau}\Big],$$
$$u = u_\infty(x, \tau) := \mathbb{E}\Big[(1-R)\,e^{-(r+x)\,\tau} - s \int_0^\tau e^{-(r+x)\,t}\,\mathrm{d}t\Big].$$

Proof
The function $p_\infty(x,\tau)$ is strictly decreasing in x, so there can be at most one solution x to this equation. Consequently, it suffices to prove that b satisfies both equations - it is then automatically the unique solution of the equation system.
Consider first the case $r+b=0$. In this case, the definition (NBD) implies $\zeta := (p-R)/c = (1-R-u)/s$. By Assumption (A), $\zeta \in (0,\infty)$, and we also observe that

$$p = c\,\zeta + R, \quad u = (1-R) - s\,\zeta.$$

Consequently, letting τ be an arbitrary positive random variable with finite mean $\zeta = \mathbb{E}[\tau] = \int_0^\tau 1\,\mathrm{d}t$, the claimed equations hold with $x=b$.
Now consider the case $r+b \neq 0$. Denoting

$$\eta := \mathbb{E}\Big[\exp\Big(-(r+b)\,\tau\Big)\Big] = \frac{p\,(r+b)-c}{R\,(r+b)-c},$$

the claimed equations are re-arranged as

$$(r+b)\,p = c + \eta\,\big(R\,(r+b)-c\big), \tag{4}$$
$$(r+b)\,u = -s + \eta\,\big((1-R)\,(r+b)+s\big). \tag{5}$$

The assumption $R\,(r+b)-c=0$ yields $p=R$, which is a contradiction to (A). Consequently, $R\,(r+b)-c \neq 0$ and Equation (4) is equivalent to $\eta = ((r+b)\,p-c)/(R\,(r+b)-c)$, which is a true statement by the assumption on τ. Plugging η into Equation (5), it is then readily verified that Equation (5) is another true statement. $\square$

The equation system in Lemma 6.2 involves model prices for the bond and the CDS - assuming they were of perpetual nature (i.e. with $T=\infty$). Now we take into account the finite maturity T. The following logic, showing the existence and uniqueness of (b_T,λ_T), is analogous to the idea in (Bernhart, Mai, 2016, Lemma 2(b)). First of all, we observe that $u_T(x,\lambda)$ is a smooth function in both variables (x,λ). Moreover,

$$\lim_{\lambda\searrow 0} u_T(x,\lambda) = -s\int_0^T e^{-(r+x)\,t}\,\mathrm{d}t, \quad \lim_{\lambda\to\infty} u_T(x,\lambda) = 1-R.$$

This makes clear that for each fixed $x \leq b_*(T)$ we can find $\lambda(x) \geq 0$ with $u = u_T(x,\lambda(x))$ by the intermediate value theorem. Furthermore, $\lambda(x)$ is unique, since it is defined as a value λ satisfying the equation

$$u + s\int_0^T e^{-(r+x+\lambda)\,t}\,\mathrm{d}t = (1-R)\int_0^T e^{-(r+x)\,t}\,\mathrm{d}\Big(1-e^{-\lambda\,t}\Big), \tag{6}$$

and the left-hand side is decreasing in λ, while the right-hand side is increasing. By the implicit function theorem, $\lambda(x)$ is even a continuous function. It is furthermore observed that $x \mapsto x+\lambda(x)$ is a non-decreasing function, cf. the induction step in the proof of (Bernhart, Mai, 2016, Lemma 2(b)). Using Equation (6), this implies that

$$g(x) := p_T(x, \lambda(x)) = \Big(c + \frac{s\,R}{1-R}\Big)\int_0^T e^{-(r+x+\lambda(x))\,t}\,\mathrm{d}t + e^{-(r+x+\lambda(x))\,T} + \frac{R\,u}{1-R} \tag{7}$$

is a decreasing function, unbounded as $x \to -\infty$. In case $b_*(T) = \infty$, we have $\lim_{x\to\infty} x + \lambda(x) = \infty$, so

$$p_T\big(b_*(T), \lambda(b_*(T))\big) = \frac{R\,u}{1-R} < p,$$

where the inequality follows like on page 5, when it was shown that the initial investment cost was positive. In case $b_*(T) < \infty$, which can only happen if $u < 0$, we observe from Equation (6) that $\lambda(b_*(T)) = 0$, so that the definition of $b_*(T)$ implies

$$p_T\big(b_*(T), \lambda(b_*(T))\big) = -\frac{c\,u}{s} + e^{-(r+b_*(T))\,T} < p,$$

where the inequality is true by Assumption (TA). Consequently, the equation $p = p_T(x, \lambda(x))$ in $x \in \mathbb{R}$ has a unique solution $\tilde{x} \in (-\infty, b_*(T))$. The pair $(b_T, \lambda_T) := (\tilde{x}, \lambda(\tilde{x}))$ is the desired, unique solution. By the implicit function theorem, the solution (b_T, λ_T) is continuous in the maturity T, since all considered functions are smooth in T.
Now let $\lambda := \frac{c-p\,(r+b)}{p-R}$. Notice that

$$\lambda > 0 \;\Leftrightarrow -\frac{c\,u}{s} < p,$$

which is a true statement by Assumption (TA). The exponential random variable τ with rate λ satisfies the assumption of Lemmata 6.1 and 6.2. Consequently, Lemma 6.2 implies that

$$p_\infty(b, \lambda) = p, \quad u_\infty(b, \lambda) = u.$$

By continuity of (b_T, λ_T) in T, it follows that $\lim_{T\to\infty}(b_T, \lambda_T) = (b, \lambda)$.

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