

THE MAXIMAL PROBABILITY FOR A JOINT DEFAULT UNDER GIVEN MARGINALS

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- Abstract Starting from given univariate survival functions, the dependence structure that maximizes the probability of a joint default can be given in closed form. This result is called "maximal coupling" or Dobrushin's Theorem in the academic literature. When the marginals are not identical, the solution is not represented by the comonotonicity copula, opposed to a modeling (mal-)practice in the financial industry. A stochastic model that respects the marginal laws and attains the upper bound for joint defaults can be extracted from the proof of the maximal coupling construction. To illustrate the theory, we bootstrap default probabilities from credit default swap contracts referencing on EU peripherals and Germany and compute the upper bound for the probability of Germany defaulting jointly with one of the peripherals.
- 1 Motivation The modeling of dependent default times is often carried out in two subsequent steps: the specification of the marginal laws and the choice of some model for the copula connecting them. Mathematically, this is justified by Sklar's theorem (see Sklar (1959)), which states that arbitrary marginals can be connected with any copula to obtain a valid joint distribution function. The main reason for the popularity of such a modeling approach is that a dependence structure can be added on top of existing, and wellunderstood, marginal models without destroying their structure. However, the danger of naïvely using this modeling paradigm is that the resulting distribution must not be reasonable with regards to the economic criterion in concern, as pointed out in the academic literature many times, see, e.g., Marshall (1996); Mikosch (2006); Genest, Nešlehová (2007); Embrechts (2009), or (Morini, 2011, Chapter 8) in the context of joint default modeling in credit risk.

A popular misbelief is that the comonotonicity copula, which maximizes the dependence if measured in terms of concordance measures, also maximizes the probability of a joint default (or, at least, the probability of default times being quite close to each other). However, this is not the case, because for two compa-

nies' default times au_1, au_2 events such as $\{| au_1- au_2|<\epsilon\}$ for small $\epsilon > 0$, or even $\{\tau_1 = \tau_2\}$, strongly depend on the marginal laws of the default times, as will be investigated in quite some detail below. We provide a simple example, which we adopt from Morini (2009, 2012): let τ_1 and τ_2 be exponentially distributed with means $1/\lambda_1 = 100$ and $1/\lambda_2 = 10$, respectively, and assume that they are coupled by a Gaussian copula C_{ρ} with parameter $\rho \in [-1, 1]$. Figure 1 visualizes¹ the probability $\rho \mapsto \mathbb{P}_{\rho}(|\tau_1 - \tau_2| < 1/12)$ that both default times happen within one month in dependence of the parameter ρ . This shows that the probability of the default times being close to each other actually decreases with increasing dependence. This might be problematic if the target risk to be modeled is not the dependence per se (being measured in terms of correlation or some more general concordance measure), but rather the probability of a joint default.



Fig. 1: The probability $\mathbb{P}_{\rho}(|\tau_1 - \tau_2| < 1/12)$ is plotted for different ρ , assuming τ_1 has an exponential distribution with $\lambda_1 = 0.01$ and τ_2 has an exponential distribution with rate $\lambda_2 = 0.1$, and these marginals are connected with a Gaussian copula with correlation parameter ρ . Notice that in the limiting case $\rho = 1$ we have $\tau_1 = \lambda_1/\lambda_2 \tau_2$ almost surely, i.e. $\mathbb{P}_1(|\tau_1 - \tau_2| < 1/12) = \mathbb{P}_1(\tau_1(1 - \lambda_1/\lambda_2) \le 1/12)$.

There are in fact various situations when the risk we truly face is the probability of a joint event, i.e. $\tau_1 = \tau_2$. Examples in a financial context are insurance portfolios with simultaneous defaults caused by a natural catastrophe, credit portfolios with some first-

$$\mathbb{P}_{\rho}(|\tau_{1} - \tau_{2}| < 1/12) = 0.001 \int_{0}^{\infty} e^{-0.01 x} \int_{\max\{0, x - 1/12\}}^{x + 1/12} c_{\rho}(1 - e^{-0.01 x}, 1 - e^{-0.1 y}) e^{-0.1 y} dy dx,$$

with c_{ρ} denoting the Gaussian copula density.

¹This probability can be evaluated numerically as a double integral:

to-default protection only, or the computation of CVA/DVA adjustments, e.g. the risk that a counterparty defaults jointly with an underlying reference entity in a credit default swap (CDS). In reliability theory, one often has two security systems with failure times τ_1, τ_2 and the risk we face is a simultaneous collapse of these systems, i.e. $\tau_1 = \tau_2$. For example, think of an energy plant which can only run when at least one of two technical security devices is working properly. When one of both components fails but the other component doesn't, then the whole system can be maintained and the failed component can be replaced as soon as possible. However, when both components fail jointly for some reason, then the whole system collapses, which is the major risk to be modeled properly. It is highly plausible that the marginal survival functions of the single components are well-understood, e.g. from information provided by their respective producers.

The present article is partly inspired by a series of papers dealing with the investigation of a multivariate distribution under given marginals but with unknown copula. The references Puccetti, Rüschendorf (2012a,b, 2013a,b) study and compute lower and upper bounds for certain functionals of a multivariate law. In comparison with these references, the present article deals with a very special functional, namely the probability of a joint default. Moreover, motivated by a financial application, Embrechts et al. (2005); Embrechts, Puccetti (2006a,b) study the Value-at-Risk and related measures of a portfolio of risks with unknown copula.

The remaining article is organized as follows: Section 2 recalls Dobrushin's Theorem on "maximal coupling" to extract an upper bound for the probability of a joint default under given marginals and Section 3 applies the result to CDS data on members of the Eurozone.

2 An upper bound for the probability of $\{\tau_1 = \tau_2\}$

To illustrate the problem, we first assume identical marginal laws, i.e. $\tau_1 \sim F$, $\tau_2 \sim F$ for a univariate distribution function F. In this case (and only in this case), coupling with the comonotonicity copula $C(u, v) = \min\{u, v\}$ indeed maximizes the probability of the event $\{\tau_1 = \tau_2\}$, implying a certain joint default, i.e. $\mathbb{P}(\tau_1 = \tau_2) = 1$. This can easily be seen from a stochastic model based on the quantile transformation: simply take $U \sim \mathcal{U}(0, 1)$ and define $\tau_1 = \tau_2 := F^{-1}(U)$, where F^{-1} is the (generalized) inverse of F. Clearly, one obtains $\mathbb{P}(\tau_1 = \tau_2) = 1$ and both default times have the pre-determined marginal law F. Conversely, $\mathbb{P}(\tau_1 = \tau_2) = 1$ already implies identical default probabilities, which follows from the fact that

$$\mathbb{P}(\tau_1 \le x) \stackrel{(*)}{=} \mathbb{P}(\tau_1 = \tau_2, \tau_1 \le x) \stackrel{(*)}{=} \mathbb{P}(\tau_2 \le x),$$

where x was arbitrary and $\mathbb{P}(\tau_1 = \tau_2) = 1$ is used in (*). This implies that for inhomogeneous marginals, there does not exist a stochastic model such that the defaults take place together for sure. Moreover, it raises the following natural question: what is the dependence structure (i.e. the copula) maximizing the probability for a joint default when the marginals are fixed? The solution to this question is given by a result called "maximal coupling" or Dobrushin's Theorem in the academic literature, see, e.g., (den

Hollander, 2010, Theorem 2.10, p. 14). The proof relies on measure theory. In the sequel, we present the proof of this result in the special case when the marginal laws have densities satisfying a mild smoothness condition. In this case, the proof requires only basic background in Analysis and therefore is hopefully more accessible to practitioners.

A probability space is constructed supporting two default times τ_1, τ_2 with given densities f_1, f_2 on $(0, \infty)$ such that the upper bound for the joint default probability is attained. To clarify notation, recall that with $F_i(x) := \int_0^x f_i(s) \, ds$, i = 1, 2, the probability of a joint default can be expressed in terms of the copula C and the marginal laws F_1, F_2 as

$$\mathbb{P}(\tau_1 = \tau_2) = \iint_{D(F_1, F_2)} dC(u, v),$$

$$D(F_1, F_2) := \{(u, v) \in [0, 1]^2 : F_1^{-1}(u) = F_2^{-1}(v)\}.$$

In order to derive the result, we need to impose a mild technical condition on the two densities:

(A) \exists some $n_0 \in \mathbb{N}$ and sequences $\{a_j^{(n)}\}_{j\in\mathbb{N}_0}$, $n \in \mathbb{N}$, partitioning $[0,\infty)$ and satisfying $0 = a_0^{(n)} < a_1^{(n)} < a_2^{(n)} < \ldots$, $\lim_{j\to\infty} a_j^{(n)} = \infty$, and $\lim_{n\to\infty} \sup_{j\in\mathbb{N}} \{a_j^{(n)} - a_{j-1}^{(n)}\} = 0$, such that for all $n \ge n_0$ on each half-open interval $(a_{j-1}^{(n)}, a_j^{(n)}]$ the densities f_1, f_2 are continuous and $f_1 - f_2$ has no sign change.

Assumption (A) is satisfied in all practical cases we can think of. An example for two densities which do not satisfy hypothesis (A) is provided in Example 2.1, clearly being pathological in nature.

Example 2.1 (Densities not satisfying (A))

Define the two non-negative functions

$$f_1(x) = 1_{\{0 < x < B_1\}} x \left| \cos \left(x^{-1} \right) \right|, \quad x > 0,$$

$$f_2(x) = 1_{\{0 < x < B_2\}} x \left| \sin \left(x^{-1} \right) \right|, \quad x > 0,$$

where $B_1, B_2 > 0$ are chosen such that f_1, f_2 are densities, i.e. integrate to 1. Then f_1, f_2 are continuous densities on the intervals $(0, B_1), (0, B_2)$ which have infinitely many points of intersection close to zero, hence they do not satisfy assumption (A).

Theorem 2.2 (A model maximizing $\mathbb{P}(au_1 = au_2)$)

Denote by C the set of all bivariate copulas and assume that τ_1, τ_2 have Riemann-integrable densities f_1, f_2 on $(0, \infty)$ satisfying assumption (A).

• One then has:

$$\sup_{C \in \mathcal{C}} \left\{ \iint_{D(F_1, F_2)} dC(u, v) \right\} = \int_0^\infty \min\{f_1(x), f_2(x)\} dx =: p.$$
 (1)

• Moreover, the supremum is actually a maximum and we can provide a probabilistic construction for the maximizer. If $f_1 = f_2$ a.e., then p = 1; if the supports of f_1 and f_2 are disjoint,

then p = 0. In all other cases we have $p \in (0, 1)$ and a maximizing copula C_{F_1,F_2} , which strongly depends on the marginals, is given by

$$C_{F_1,F_2}(u,v) = \int_0^{\min\{F_1^{-1}(u),F_2^{-1}(v)\}} \min\{f_1(s),f_2(s)\} ds$$

+ $\frac{1}{1-p} \left(\int_0^{F_1^{-1}(u)} f_1(s) - \min\{f_1(s),f_2(s)\} ds \right) \times$
 $\times \left(\int_0^{F_2^{-1}(v)} f_2(s) - \min\{f_1(s),f_2(s)\} ds \right).$

Proof

For the sake of notational simplicity, we assume that the sequences $\{a_j^{(n)}\}_{j\in\mathbb{N}_0}$ in assumption (A) are given by $a_j^{(n)} = j/n$. This excludes irrational points of a sign change of $f_1 - f_2$. The general case, however, is derived analogously, only requiring a more involved notation. We first consider two degenerate cases:

- If the supports of f_1 and f_2 are disjoint, then obviously we observe $\mathbb{P}(\tau_1 = \tau_2) = 0$, irrespectively of their copula.
- If $f_1 = f_2$ a.e., then the distributions of τ_1 and τ_2 are identical and the comonotonicity copula $\min\{u, v\}$ provides the maximum $\mathbb{P}(\tau_1 = \tau_2) = 1$.

We define $p := \int_0^\infty \min\{f_1(x), f_2(x)\}dx$, which – excluding the two degenerate cases from above – is in (0, 1). Moreover, define the densities

$$h_{min} := \frac{1}{p} \min\{f_1, f_2\}, \quad h_{f_1} := \frac{1}{1-p} (f_1 - p h_{min}),$$
$$h_{f_2} := \frac{1}{1-p} (f_2 - p h_{min}).$$

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the independent random variables $H_{min} \sim h_{min}$, $H_{f_1} \sim h_{f_1}$, $H_{f_2} \sim h_{f_2}$, and X a Bernoulli variable with success probability p. Define

$$(\tau_1, \tau_2) := (H_{min}, H_{min}) \mathbf{1}_{\{X=1\}} + (H_{f_1}, H_{f_2}) \mathbf{1}_{\{X=0\}}.$$

It is now easy to verify that $\mathbb{P}(\tau_1=\tau_2)=p$ and

$$\mathbb{P}(\tau_i \le x) = p \,\mathbb{P}(H_{min} \le x) + (1-p) \,\mathbb{P}(H_{f_i} \le x) \\ = \int_0^x p \,h_{min}(s) + (1-p) \,h_{f_i}(s) ds = \int_0^x f_i(s) ds = F_i(x),$$

for i = 1, 2. Left to check is that p is actually an upper bound for $\mathbb{P}(\tau_1 = \tau_2)$ across all possible copulas $C \in \mathcal{C}$. This can be observed by discretizing the probability law. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting (τ_1, τ_2) with given marginals and arbitrary copula C. For each $n \in \mathbb{N}$ denote the ceiling function by $\lceil \cdot \rceil$ and

$$\left(\tau_1^{(n)}, \tau_2^{(n)}\right) := \left(\frac{\lceil n \, \tau_1 \rceil}{n}, \frac{\lceil n \, \tau_2 \rceil}{n}\right) \in \left(\frac{1}{n} \, \mathbb{N}\right)^2.$$

Fix $j \in \mathbb{N}$. Since $\{\tau_1^{(n)} = \tau_2^{(n)} = j/n\} \subset \{\tau_i^{(n)} = j/n\}$ for i = 1, 2, we obtain the inequality

$$\mathbb{P}(\tau_1^{(n)} = \tau_2^{(n)} = j/n) \le \min\left\{\mathbb{P}(\tau_1^{(n)} = j/n), \mathbb{P}(\tau_2^{(n)} = j/n)\right\}$$
$$= \min\left\{\int_{\frac{j-1}{n}}^{\frac{j}{n}} f_1(x) \, dx, \int_{\frac{j-1}{n}}^{\frac{j}{n}} f_2(x) \, dx\right\}.$$

The piecewise continuity assumption on the densities allows us to apply the mean-value theorem of integration, which provides within each ((j-1)/n, j/n] some values $\xi_{j,n}$, $\eta_{j,n}$ with

$$\frac{1}{n}f_1(\xi_{j,n}) = \int_{\frac{j-1}{n}}^{\frac{2}{n}} f_1(x) \, dx, \quad \frac{1}{n}f_2(\eta_{j,n}) = \int_{\frac{j-1}{n}}^{\frac{2}{n}} f_2(x) \, dx.$$
 (2)

On the interval ((j-1)/n, j/n] we have by hypothesis (A) either $f_2 \leq f_1$ or $f_1 \leq f_2$. By monotonicity of the integral, we have the case $f_1 \leq f_2$ if and only if

$$f_1(\xi_{j,n}) = n \, \int_{\frac{j-1}{n}}^{\frac{j}{n}} f_1(x) \, dx \le n \, \int_{\frac{j-1}{n}}^{\frac{j}{n}} f_2(x) \, dx = f_2(\eta_{j,n}).$$

Hence, by defining

$$x_{j,n} := \begin{cases} \xi_{j,n}, & f_1(\xi_{j,n}) < f_2(\eta_{j,n}) \\ \eta_{j,n}, & f_1(\xi_{j,n}) \ge f_2(\eta_{j,n}) \end{cases},$$

we have that

$$\min\{f_1(x_{j,n}), f_2(x_{j,n})\} = \min\{f_1(\xi_{j,n}), f_2(\eta_{j,n})\}.$$
 (3)

This implies that

$$\mathbb{P}(\tau_{1} = \tau_{2}) \leq \mathbb{P}\big(\cap_{n \in \mathbb{N}} \big\{\tau_{1}^{(n)} = \tau_{2}^{(n)}\big\}\big) = \lim_{n \to \infty} \mathbb{P}\big(\tau_{1}^{(n)} = \tau_{2}^{(n)}\big) \\
= \lim_{n \to \infty} \sum_{j \in \mathbb{N}} \mathbb{P}\big(\tau_{1}^{(n)} = \tau_{2}^{(n)} = j/n\big) \\
\leq \lim_{n \to \infty} \frac{1}{n} \sum_{j \in \mathbb{N}} \min\{f_{1}(\xi_{j,n}), f_{2}(\eta_{j,n})\} \\
\stackrel{(3)}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{j \in \mathbb{N}} \min\{f_{1}(x_{j,n}), f_{2}(x_{j,n})\} \\
\stackrel{(*)}{=} \int_{0}^{\infty} \min\{f_{1}(x), f_{2}(x)\} dx,$$

establishing the claim. Here, (*) holds due to Riemann-integrability of $x \mapsto \min\{f_1(x), f_2(x)\}$. Finally, a straightforward computation shows that the maximizing copula is of the claimed form. \Box

Remark 2.3 (The mean value theorem of integration)

In the proof of Theorem 2.2 we made use of the mean value theorem in (2). The classical version is formulated for continuous functions on a compact interval, which are therefore bounded. However, we applied a slight generalization for half-open and bounded intervals, because we want to allow the densities to have poles. For instance, many standard densities, such as certain Gamma distributions, have a pole at zero. However, this

required generalization is straightforward. Since we could not find a reference, we provide a short proof in the sequel: Let $f:(a,b] \to [0,\infty)$ be continuous and assume that the integral $\int_a^b f(x) dx$ is finite, however we might have $\lim_{x\downarrow a} f(x) = \infty$. For each $n \in \mathbb{N}$ we apply the classical mean value theorem on the compact interval [a + 1/n, b] providing numbers $\xi_n \in (a, b]$ with

$$\left(b-a-\frac{1}{n}\right)f(\xi_n) = \int_{a+\frac{1}{n}}^b f(x)\,dx.$$

Since the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded, we find a convergent subsequence $\{\xi_{n_k}\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\xi_{n_k} =: \xi \in [a,b]$. Using continuity of f we find that

$$(b-a) f(\xi) = \int_a^b f(x) \, dx,$$

as desired. Finally, note that if $\lim_{x\downarrow a} f(x) = \infty$ we must have $\xi > a$, since the integral was assumed to be finite.

Remark 2.4 (Finite time horizon)

Since a derivative contract typically has a finite maturity T, one might rather be interested in the probability that a joint default happens during the lifetime of the contract, i.e. one is interested in the event $\{\tau_1 = \tau_2 \leq T\}$. It follows immediately from the proof above that the probability $\mathbb{P}(\tau_1 = \tau_2 \leq T)$ is maximized by the same stochastic model (respectively the given copula), and the upper bound is given by

$$\mathbb{P}(\tau_1 = \tau_2 \le T) \le \int_0^T \min\{f_1(x), f_2(x)\} dx.$$

3 A Eurozone case study We bootstrap survival probabilities from credit default swaps (CDS) referring to senior debt issued by certain members of the Eurozone, namely by Germany, Greece, and Portugal. This is accomplished by the common market practice of assuming piecewise constant default intensities, and iteratively matching market-observed CDS prices with increasing maturity, as described, e.g., in O'Kane, Turnbull (2003). Mathematically, the default times of Germany, Greece, and Portugal are denoted by τ_{Ger}, τ_{Gre} , and τ_{Por} , respectively. For $* \in \{Ger, Gre, Por\}$, the density of τ_* is assumed to be given by

$$f_*(t) = \lambda_*(t) \exp\left(-\int_0^t \lambda_*(s) \, ds\right), \quad t \ge 0,$$
$$\lambda_*(x) = \sum_{i=1}^5 \lambda_*^{(i)} \, \mathbb{1}_{\{t \in [i-1,i)\}} + \lambda_*^{(7)} \, \mathbb{1}_{\{t \in [5,7)\}} + \lambda_*^{(\infty)} \, \mathbb{1}_{\{t \in [7,\infty)\}},$$

where the non-negative default intensities $\lambda_*^{(1)}, \ldots, \lambda_*^{(7)}, \lambda_*^{(\infty)}$ are bootstrapped iteratively from quoted CDS spreads with maturities 1, 2, 3, 4, 5, 7, 10 years. These densities satisfy hypothesis (A) of Theorem 2.2. Figure 2 visualizes the survival functions we obtain using this approach. One observes that $\bar{F}_{Ger} > \bar{F}_{Por} > \bar{F}_{Gre}$ pointwise, which mirrors the market's currently prevailing

opinion that Germany is a solid debtor, an investment into Portuguese debt is risky, and into Greek debt audacious. Note that this already implies $\mathbb{P}(\tau_i = \tau_j) = 0$ if τ_i and τ_j are connected with the comonotonicity copula, $i, j \in \{Ger, Gre, Por\}$.



Fig. 2: Market-implied survival functions for Germany, Greece, and Portugal. These data are bootstrapped from CDS quotes published via Bloomberg on December 12, 2012.

Consequently, the probability that Germany defaults jointly with Portugal (resp. Greece) is naturally bounded from above. To be precise, the probability that Germany defaults jointly with Portugal is at most 47% in any model that is consistent with market CDS quotes. The probability that Germany defaults jointly with Greece is even bounded from above by 14% in all models that are consistent with CDS quotes. Figure 3 shows scatterplots from the "worst case" copulas $C_{F_{Ger},F_{Por}}$, respectively $C_{F_{Ger},F_{Gre}}$, which attain these upper bounds. As one can observe, in this "worst case" model it is even impossible that Germany defaults before Portugal or Greece.

- **4 Conclusion** Given two default times τ_1, τ_2 with non-identical marginal laws on $[0, \infty)$, we have illustrated that it is not the comonotonicity copula that maximizes the probability of the event $\{\tau_1 = \tau_2\}$, correcting a dangerous misbelief in the financial industry. Even worse, a maximizing copula necessarily depends on the given marginal laws. We have computed an upper bound for the probability of a joint failure and have shown that this upper bound is sharp. It is attained by a probabilistic model which we have constructed explicitly, and for which we could compute the copula which highly depends on the given marginals in closed form.
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Fig. 3: Scatterplots from the market-implied "worst-case" copulas which maximize the probability of a joint default of Germany and Portugal/Greece. The simulation has been achieved based on the stochastic model derived in the proof of Theorem 2.2, where the required random variables were simulated via their densities using the obvious acceptance-rejection algorithms, see, e.g., (Korn et al., 2010, p. 33) for detailed information on the latter.