THE NON-CENTRAL χ^2-LAW IN FINANCE: A SURVEY

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Abstract The non-central χ^2-distribution appears in various financial applications. The present article surveys the key features to be known about this distribution, reviews some numerical techniques involved in its implementation, and collects a number of examples for its use in the context of Mathematical Finance. The article is organized as follows: Section 1 defines the non-central χ^2-law and reviews its key features. Section 2 defines squared Bessel processes, which might be viewed as dynamic counterparts of the static non-central χ^2-distribution, and surveys their key features. Finally, Section 3 briefly outlines some of the financial applications involving the non-central χ^2-distribution, and Section 4 concludes.

1 Static viewpoint: the non-central χ^2-distribution

Throughout, we denote by

$$||\boldsymbol{x}|| := \sqrt{x_1^2 + \ldots + x_\delta^2}, \quad \boldsymbol{x} = (x_1, \ldots, x_\delta) \in \mathbb{R}^\delta$$

the Euclidean norm on $\mathbb{R}^\delta$. If $\boldsymbol{Z}$ is multivariate normal in dimension $\delta \in \mathbb{N}$ with mean vector $\boldsymbol{\mu} \in \mathbb{R}^\delta$ and the identity matrix as covariance matrix, then the random variable $Y := ||\boldsymbol{Z}||^2$ is said to have a non-central χ^2-law with non-centrality parameter $\alpha := ||\boldsymbol{\mu}||^2 \geq 0$ and δ degrees of freedom. This is the most common and most intuitive introduction of the non-central χ^2-law. In particular, if $\alpha = 0$, one is left with the regular χ^2-distribution with $\delta \in \mathbb{N}$ degrees of freedom, which is well-known from elementary statistics lectures. For many applications in Mathematical Finance it is reasonable to generalize the definition also to non-integer degrees of freedom $\delta > 0$. This is formally accomplished in the following definition.

Definition 1.1 (The non-central χ^2-distribution)
A positive random variable Y is said to follow the *non-central χ^2-distribution* with $\delta > 0$ degrees of freedom and non-centrality parameter $\alpha > 0$, denoted $Y \sim \chi^2(\delta, \alpha)$, if its density is given by

$$f_{\chi^2(\delta,\alpha)}(x) := \frac{1}{2}\, e^{-\frac{\alpha+x}{2}} \left(\frac{x}{\alpha}\right)^{\frac{\delta}{4}-\frac{1}{2}} I_{\frac{\delta}{2}-1}\big(\sqrt{\alpha\, x}\big), \quad x > 0,$$

where I_ν denotes the modified Bessel function of the first kind with index ν, given by the series representation

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{\nu+2\,n}, \quad x > 0.$$

An efficient evaluation of the function $I_\nu(x)$ for real ν and non-negative x, as required, is standard. For instance, in MATLAB it is pre-implemented in the routine `besseli`. Consequently, the density of the non-central χ^2-distribution is given in a numerically convenient closed form.

1.1 Laplace transform

A straightforward computation shows that the Laplace transform of $Y \sim \chi^2(\delta,\alpha)$ is given by

$$\mathbb{E}\Big[e^{-u\,Y}\Big] = \frac{1}{(1+2\,u)^{\frac{\delta}{2}}}\,e^{-\frac{\alpha\,u}{1+2\,u}} =: e^{-\Psi(u)}, \quad u \geq 0,$$
$$\Psi(u) = \frac{\delta}{2}\,\log(1+2\,u) + \frac{\alpha\,u}{1+2\,u}, \quad u \geq 0. \tag{1}$$

The same computation (with $u=0$) indeed also verifies that $f_{\chi^2(\delta,\alpha)}$ is a proper density. The function $\Psi : [0,\infty) \to [0,\infty)$ is a so-called *complete Bernstein function*, see Schilling et al. (2010) for detailed information on these functions. This implies that the non-central χ^2-law is infinitely divisible, and even more, it is part of the so-called *Bondesson family* of distributions, which was introduced in Bondesson (1981) under the name g.c.m.e.d. distributions, see also Bondesson (1979). The latter distributions arise, e.g., as the laws of first hitting times of diffusion processes and have very convenient properties, see Schilling et al. (2010) for details. Moreover, we observe that $\Psi = \Psi_1 + \Psi_2$, where

$$\Psi_1(u) = \frac{\delta}{2}\,\log(1+2\,u), \quad \Psi_2(u) = \frac{\alpha\,u}{1+2\,u}, \quad u \geq 0,$$

are also two complete Bernstein functions. The Bernstein function Ψ_1 corresponds to a Γ-distribution $\Gamma(\delta/2, 1/2)$, using the parameterization of (Mai, Scherer, 2012, p. 2). The Bernstein function Ψ_2 corresponds to a compound Poisson sum (the Poisson counting variable has mean $\alpha/2$) of exponentially distributed random variables with mean 2.

Remark 1.2 (Zero degrees of freedom)
The function Ψ is clearly also a complete Bernstein function in the case $\delta = 0$ (pure compound Poisson case). Hence, it makes sense to define the non-central χ^2-law for all $\delta \geq 0$, but for $\delta = 0$ it has no density, because it has an atom at zero.

1.2 Mixture representation

It is readily observed that

$$f_{\chi^2(\delta,\alpha)}(x) = \sum_{n=0}^{\infty} \frac{e^{-\frac{\alpha}{2}}}{n!}\,\Big(\frac{\alpha}{2}\Big)^n\,f_{\Gamma(\delta/2+n,1/2)}(x), \quad x > 0, \tag{2}$$

where $f_{\Gamma(\beta,\eta)}$ denotes the density of a Γ-distribution with parameters β,η, again using the parameterization of (Mai, Scherer, 2012, p. 2). This shows that the non-central χ^2-distribution may be viewed as the Poisson mixture of Gamma distributions. Concerning a stochastic representation, it follows from the aforementioned decomposition of Ψ into Ψ_1 and Ψ_2 that

$$Y := G + \sum_{k=1}^{N} J_k \sim \chi^2(\delta,\alpha), \tag{3}$$

where[1] $G \sim \Gamma(\delta/2, 1/2)$, $N \sim$ Poi$(\alpha/2)$ and $J_1, J_2, \ldots \sim \mathcal{E}(1/2)$ are stochastically independent. This stochastic representation implies a quick simulation algorithm for $\chi^2(\delta, \alpha)$, provided simulation algorithms for the involved Gamma-, Poisson-, and exponential distribution are given. For the latter, see (Mai, Scherer, 2012, Chapter 6). Recalling the well-known fact that $\Gamma(\delta/2, 1/2)+\Gamma(N, 1/2) = \Gamma(\delta/2 + N, 1/2)$, this is in line with the Poisson mixture representation (2) of the density, respectively an alternative derivation thereof. Alternatively, one might also re-write the sum in (3) as a stopped Gamma subordinator[2]:

$$\sum_{k=1}^{N} J_k \stackrel{d}{=} \Lambda_N,$$

where $\Lambda = \{\Lambda_t\}_{t\geq 0}$ is a Gamma subordinator[3], independent of $N \sim$ Poi$(\alpha/2)$, with parameters $(1, 1/2)$, using the parameterization of (Mai, Scherer, 2012, p. 274).

1.3 Distribution function

Starting from the representation (2), the distribution function is a Poisson mixture of certain Gamma distributions. Benton, Krishnamoorthy (2003) propose an algorithm to compute this series efficiently[4]. Alternatively[5], we present a way of computing the distribution function based on (Bernhart et al., 2014, Corollary 3.4). This method to compute the distribution function relies on Laplace inversion, i.e. the distribution function is retrieved from its known Laplace transform. It is applicable in the present situation, since the non-central χ^2-distribution is part of the Bondesson class, for which Bernhart et al. (2014) is known to work. If we denote the distribution function of $Y \sim \chi^2(\delta, \alpha)$ by $F_{\chi^2(\delta,\alpha)}$, (Bernhart et al., 2014, Corollary 3.4) implies for $x > 0$ that $F_{\chi^2(\delta,\alpha)}(x) =$

$$\frac{M}{\pi} \int_0^1 \text{Im}\left(\frac{e^{x\,a - x\,M\,\log(v)\,(b\,i-a) - \Psi\left(a - M\,\log(v)\,(b\,i-a)\right)}}{a/(b\,i-a) - M\,\log(v)}\right) \frac{dv}{v}, \quad (4)$$

for arbitrary $a, b > 0$ and $M > 2/(a\,x)$. The integrand vanishes as $v \searrow 0$, i.e. the integral is a proper Riemann integral. In particular, choosing $a = b = 1/x$, and $M > 2$ arbitrary, the integrand in (4) can be bounded from above by a constant independent of x and δ as follows:

$$\left|\text{Im}\left(\frac{e^{x\,a - x\,M\,\log(v)\,(b\,i-a) - \Psi\left(a - M\,\log(v)\,(b\,i-a)\right)}}{a/(b\,i-a) - M\,\log(v)}\right)\right| \leq 2\,v^M\,e^{1+\frac{\alpha}{4}}.$$

[1] We denote by Poi(λ) and $\mathcal{E}(\lambda)$ the Poisson- and exponential distribution with rate parameter $\lambda > 0$.

[2] We denote by $\stackrel{d}{=}$ equality in distribution.

[3] Recall that a Gamma subordinator $\{\Lambda_t\}_{t\geq 0}$ with parameters (β, η) is a non-decreasing stochastic process with $\Lambda_0 = 0$, independent and stationary $\Gamma(t\,\beta, \eta)$-distributed increments $\Lambda_{t+s} - \Lambda_s$, $0 \leq s < t$.

[4] See also Dalgaard (2001) and the references cited in Benton, Krishnamoorthy (2003) for related literature.

[5] The Laplace inversion formula (4) agrees with the method described by Benton, Krishnamoorthy (2003), which is also pre-implemented in MATLAB as `ncx2cdf`, up to an accuracy of about 10^{-12}. We tested several sets of parameters α, δ.

Interestingly, there exists also a quick-and-dirty approximation for the distribution function, which might be useful for an approximative implementation on an EXCEL spreadsheet. It is derived in Sankaran (1959) and states for $x \geq 0$ that $F_{\chi^2(\delta,\alpha)}(x) \approx$

$$\Phi\left(\frac{\left(\frac{x}{\alpha+\delta}\right)^{h_{\delta,\alpha}} - \left(1 + h_{\delta,\alpha}\, p_{\delta,\alpha}\, \left(h_{\delta,\alpha} - 1 - \frac{(2-h_{\delta,\alpha})\, m_{\delta,\alpha}\, p_{\delta,\alpha}}{2}\right)\right)}{h_{\delta,\alpha}\, \sqrt{2\, p_{\delta,\alpha}}\, (1 + m_{\delta,\alpha}\, p_{\delta,\alpha}/2)} \right), \tag{5}$$

where Φ denotes the standard normal cdf and

$$h_{\delta,\alpha} = 1 - \frac{2\,(\delta+\alpha)\,(\delta+3\,\alpha)}{3\,(\delta+2\,\alpha)^2}, \quad p_{\delta,\alpha} = \frac{\delta + 2\,\alpha}{(\delta+\alpha)^2},$$
$$m_{\delta,\alpha} = (h_{\delta,\alpha} - 1)\,(1 - 3\,h_{\delta,\alpha}).$$

Figure 1 visualizes this approximation for $\alpha = 10$ and $\delta = 0.1$.

1.4 (Truncated) moments

For $Y \sim \chi^2(\delta, \alpha)$ we turn our attention to the task of computing $\mathbb{E}[Y^p]$, $\mathbb{E}[Y^p\, 1_{\{Y>k\}}]$, and $\mathbb{E}[Y^p\, 1_{\{Y<k\}}]$ for $k > 0$ and $p \geq 0$. Notice that all moments exist, since the Laplace transform can be extended to the domain $(-1/2, \infty)$, i.e. a domain including a neighborhood of zero. For arbitrary $p > -\delta/2$, the following formula is derived in Carr, Linetsky (2006):

$$\mathbb{E}[Y^p] = 2^p\, e^{-\frac{\alpha}{2}}\, \frac{\Gamma(p + \delta/2)}{\Gamma(\delta/2)}\, {}_1F_1(p + \delta/2, \delta/2, \alpha/2), \tag{6}$$

where ${}_1F_1$ denotes the Kummer confluent hypergeometric function, given by

$${}_1F_1(a, b, x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, \frac{\prod_{i=0}^{n-1} (a+i)}{\prod_{i=0}^{n-1} (b+i)}.$$

The latter function is not always easy to evaluate numerically, depending on its arguments. A survey of different numerical approaches to implement it can be found in (Pearson, 2009, Chapter 3). For $p \in \mathbb{N}_0$ the regular moments can be computed conveniently according to the following recursion:

$$\mathbb{E}[Y^0] = 1,\ \mathbb{E}[Y^p] = \sum_{k=0}^{p-1} \frac{(p-1)!}{k!}\, 2^{p-k-1}\, \big(\delta + \alpha\,(p-k)\big)\, \mathbb{E}[Y^k].$$

This recursion is easily obtained by computing the derivatives of the Laplace transform and the Laplace exponent at zero. It is convenient and robust to implement. For arbitrary $p > -\delta/2$ and $k \geq 0$, the following formulas for the truncated moments are also derived in Carr, Linetsky (2006):

$$\mathbb{E}[Y^p\, 1_{\{Y \leq k\}}] = 2^p \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}} \left(\frac{\alpha}{2}\right)^n \frac{\gamma(\delta/2 + p + n, k/2)}{n!\, \Gamma(\delta/2 + n)},$$
$$\mathbb{E}[Y^p\, 1_{\{Y > k\}}] = 2^p \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}} \left(\frac{\alpha}{2}\right)^n \frac{\Gamma(\delta/2 + p + n, k/2)}{n!\, \Gamma(\delta/2 + n)},$$

where $\gamma(a, x) = \int_0^x y^{a-1}\, e^{-y}\, dy$ and $\Gamma(a, x) = \Gamma(x) - \gamma(a, x)$ denote incomplete Gamma functions. The function $\gamma(a, x)/\Gamma(a)$

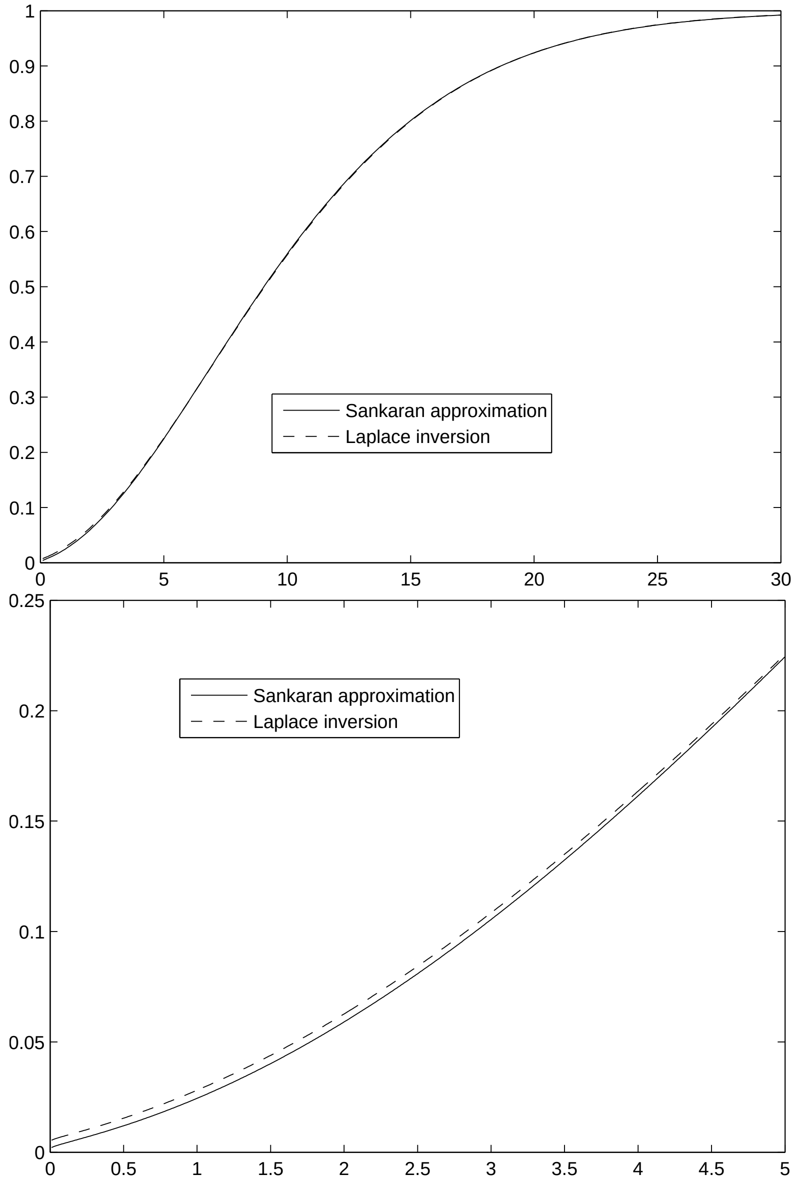

Fig. 1: Visualization of $F_{\chi^2(0.1,10)}(x)$, computed from Formula (4) which is based on Laplace inversion, in comparison with its Sankaran approximation (5). The bottom plot highlights evaluations for arguments $x \in (0, 5)$, where the Sankaran approximation lacks precision. It is observed that the Sankaran approximation might be okay for rough applications, but should not be used in applications requiring high precision.

is pre-implemented in MATLAB as the routine `gammainc`. Notice that for $p = 0$ these formulas give an alternative representation for the distribution function, which was treated in the previous paragraph. As a final remark, the algorithm derived in Benton, Krishnamoorthy (2003) can be adapted easily to evaluate not only the distribution function but also the truncated moments efficiently. The underlying idea is to truncate the sums in a tricky way by starting at an index n corresponding to the mean of the involved Poisson mixing variable and adding up the summands $n-1, n-2, \ldots$ and $n+1, n+2, \ldots$ until their contribution to the overall sum becomes negligible.

2 Dynamic viewpoint: squared Bessel processes

The non-central χ^2-distribution arises as the marginal law of so-called squared Bessel processes. In this sense, squared Bessel processes may be viewed as a dynamic counterpart to the non-central χ^2-distribution, just like Brownian motion is a dynamic counterpart of the standard normal distribution. In Mathematical Finance there are many applications involving squared Bessel processes, and hence implicitly the non-central χ^2-distribution. Therefore, we collect some useful background information regarding Bessel processes in the present section. To this end, we denote by $W = \{W_t\}$ standard Brownian motion.

Definition 2.1 ((Squared) Bessel process)

(a) A diffusion process $\{X_t\}$ is called a *Bessel process with index* $\nu \in \mathbb{R}$ if it follows the stochastic differential equation

$$\mathrm{d}X_t = \frac{2\,\nu + 1}{2\,X_t}\,\mathrm{d}t + \mathrm{d}W_t.$$

(b) A diffusion process $\{Y_t\}$ is called *squared Bessel process of dimension* δ if there exists a Bessel process $\{X_t\}$ of index $\nu = (\delta - 2)/2$ such that $\{Y_t\} \overset{d}{=} \{X_t^2\}$, or equivalently, if it follows the stochastic differential equation

$$\mathrm{d}Y_t = \delta\,\mathrm{d}t + 2\,\sqrt{|Y_t|}\,\mathrm{d}W_t.$$

Notice in particular that the absolute value under the square root in the defining stochastic differential equation of the squared Bessel process may be dropped a posteriori in the case $\delta \geq 0$, because $Y_t \geq 0$ almost surely in this case, see Subsection 2.2 below.

2.1 Existence of (squared) Bessel processes

If $\delta := 2\,\nu + 2 \in \{2, 3, 4, \dots\}$ is a natural number ≥ 2, then it can be shown using Itô calculus that $X_t := ||\boldsymbol{\mu} + W_t^{(\delta)}||$, $t \geq 0$, is a Bessel process with index ν, where $\{W_t^{(\delta)}\}$ denotes standard, δ-dimensional Brownian motion, and $\boldsymbol{\mu} \in \mathbb{R}^\delta$. Hence, in this case a squared Bessel process of dimension δ exists, namely $Y_t := X_t^2 = ||\boldsymbol{\mu} + W_t^{(\delta)}||^2$, $t \geq 0$. For $t > 0$ we observe

$$\begin{aligned}\frac{Y_t}{t} &= \Big|\Big|\frac{\boldsymbol{\mu}}{\sqrt{t}} + \frac{W_t^{(\delta)}}{\sqrt{t}}\Big|\Big|^2 \\ &\overset{d}{=} \Big|\Big|\frac{\boldsymbol{\mu}}{\sqrt{t}} + W_1^{(\delta)}\Big|\Big|^2 \sim \chi^2\Big(\delta,\, \frac{||\boldsymbol{\mu}||^2}{t}\Big) = \chi^2\Big(\delta,\, \frac{X_0^2}{t}\Big).\end{aligned}$$

It follows that the Markov transition kernel of the squared Bessel process of dimension δ, i.e. the conditional density of Y_{s+t} given $Y_s = x$ for arbitrary $s \geq 0$, is given by

$$p_t(x, y) := \frac{1}{t}\, f_{\chi^2(\delta, x/t)}\Big(\frac{y}{t}\Big), \quad t, x, y > 0.$$

Considering the Markov transition kernel for $\delta \in \{2, 3, 4, \dots\}$, one may define an analytical extension of it to arbitrary degrees of freedom $\delta \geq 0$, just like the non-central χ^2-law is extended analytically to $\delta \geq 0$. With a Kolmogorov existence argument it can be shown that a squared Bessel process of arbitrary dimension $\delta \geq 0$ exists. Hence, a Bessel process of arbitrary index $\nu \geq -1$ exists and is defined as the square root of the squared Bessel process constructed in this way. It is further explained in, e.g., Göing-Jaeschke, Yor (2003) that Bessel processes with arbitrary real index $\nu \in \mathbb{R}$ and squared Bessel processes of arbitrary dimension $\delta \in \mathbb{R}$ exist as well, i.e. the defining stochastic differential equations admit unique solutions, which are adapted to the filtration of the driving Brownian motion (so-called strong solution). Squared Bessel processes with negative index that start at a positive value almost surely hit zero in finite time. After that they become negative and behave similar to the negative of a squared Bessel process with positive index.

2.2 Hitting times of zero

The following can be shown[6] for a squared Bessel process $Y = \{Y_t\}$ of dimension $\delta \geq 0$ which starts at a positive value $Y_0 > 0$, which of course translates into analogous statements for Bessel processes with index $\nu := (\delta - 2)/2 \geq -1$:

- $\underline{\delta = 0}$: The process Y hits zero almost surely in finite time, and remains there.
- $\underline{0 < \delta < 2}$: The process Y hits zero infinitely many times and $\limsup_{t\to\infty} Y_t = \infty$.
- $\underline{\delta = 2}$: The process Y is strictly positive with $\liminf\limits_{t\to\infty} Y_t = 0$ and $\limsup_{t\to\infty} Y_t = \infty$.
- $\underline{\delta > 2}$: The process Y is strictly positive with $\lim_{t\to\infty} Y_t = \infty$.

Notice in particular that these statements contain the well-known fact that the origin (or any other point) is recurrent for one- and two-dimensional Brownian motion, whereas it is transient in larger dimensions. It is further easy to see from the Laplace transform of the non-central χ^2-law that for $\delta = 0$ we have

$$\mathbb{P}(Y_t = 0) = \lim_{u\to\infty} \mathbb{E}\Big[e^{-u\,Y_t}\Big] = \lim_{u\to\infty} e^{-\frac{Y_0\,u}{1+2\,u\,t}} = e^{-\frac{Y_0}{2\,t}}, \quad t > 0.$$

2.3 Further useful properties

One immediately checks that a squared Bessel process $Y = \{Y_t\}$ satisfies the *scaling property* $\{Y_t\} \stackrel{d}{=} \{Y_{c\,t}/c\}$ for each $c > 0$. Another key feature is the *additivity property*, stating that the sum of independent squared Bessel processes of non-negative dimensions $\delta_1, \delta_2 \geq 0$ is again a squared Bessel process of dimension $\delta_1 + \delta_2$, see Shiga, Watanabe (1973). Moreover, one of the most useful identities regarding Bessel processes relates its law to the distribution of a Bessel process with index zero. This is especially useful when dealing with Bessel processes with negative index (which might diffuse to zero), because it allows to transform the analysis to a Bessel process which cannot diffuse to zero. Recall from Subsection 2.2 that a Bessel process remains almost surely positive whenever its index ν is non-negative. The following lemma is stated and massively used in Yor (1992).

[6] See, e.g., http://almostsure.wordpress.com/2010/07/28/bessel-processes/.

Lemma 2.2 (Useful transformation of Bessel index)
On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ supporting a standard Brownian motion $\{W_t\}$ with natural filtration $\{\mathcal{F}_t\}$, denote the Bessel process with index $\nu \in \mathbb{R}$ by $X^{(\nu)} = \{X_t^{(\nu)}\}$, and its first hitting time of zero by $T_\nu \in (0, \infty]$. Moreover, we denote by $X^{(0)} = \{X_t^{(0)}\}$ the Bessel process with index zero and $X_0^{(0)} = X_0^{(\nu)} > 0$. For all bounded, continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $t > 0$ it holds that

$$\mathbb{E}\Big[f\big(X_t^{(\nu)}\big)1_{\{T_\nu > t\}}\Big] = \mathbb{E}\left[f\big(X_t^{(0)}\big)\left(\frac{X_t^{(0)}}{X_0^{(0)}}\right)^{\nu} e^{-\frac{\nu^2}{2}\int_0^t \frac{1}{X_s^{(0)}}\mathrm{d}s}\right].$$

Remark 2.3 (Bessel processes and Asian options)
Lemma 2.2 is used in Yor (1992) in order to derive the density of the integral over geometric Brownian motion, which plays a dominant role in Asian option pricing within a Black-Scholes cosmos. The reference Carr, Schröder (2004) further explains in quite some detail how Bessel processes play a dominant role in this context.

3 Applications in Mathematical Finance

The popularity of the non-central χ^2-distribution in Mathematical Finance stems from its involvement in two quite popular models: the CEV stock price model and the CIR model. The underlying stochastic processes in both models can be viewed as certain transformations of squared Bessel processes, indicating how the non-central χ^2-distribution comes into play.

3.1 The CEV model

The *constant elasticity of variance (CEV)* model defines the price process $\{S_t\}_{t\geq 0}$ of an asset as a diffusion process satisfying the stochastic differential equation

$$\mathrm{d}S_t = \mu\, S_t\,\mathrm{d}t + \sigma\, S_t^{\rho}\,\mathrm{d}W_t, \tag{7}$$

where the parameter $\rho \geq 0$ controls the relation between level and volatility of the price process. For $\rho < 1$, the volatility increases with a decreasing asset price (leverage effect), whereas it increases with an increasing asset price for $\rho > 1$ (inverse leverage effect). For $\rho = 1$, the model boils down to the standard Black-Scholes model with volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$. In the leverage effect case $\rho \in (0, 1)$ and for non-negative $\mu \geq 0$, the reference Delbaen, Shirakawa (2002) interprets S as a stock price process and uses Lemma 2.2 in order to show that the process $\{S_t\}_{t\geq 0}$, given by

$$S_t := e^{\mu\, t}\left(Y^{(\delta_\rho)}_{\min\{T_{\delta_\rho}, \frac{\sigma^2\,(1-\rho)}{2\,\mu}\,\big(1-e^{-2\,\mu\,(1-\rho)\,t}\big)\}}\right)^{\frac{1}{2-2\,\rho}},$$

satisfies (7), where $\delta_\rho := (1-2\,\rho)/(1-\rho) \in (-\infty, 1)$, $Y^{(\delta_\rho)} = \{Y_t^{(\delta_\rho)}\}_{t\geq 0}$ is a squared Bessel process of dimension δ_ρ starting at $Y_0^{(\delta_\rho)} = S_0^{2-2\,\rho}$, and $T_{\delta_\rho} \in (0, \infty]$ its first hitting time of zero. This stochastic representation of the CEV process in terms of a squared Bessel process allows to derive its probability distribution in terms of the non-central χ^2-distribution:

- For non-negative $x \geq 0$ it follows with the help of Lemma 2.2 that $\mathbb{P}(S_t > x) =$

$$F_{\chi^2\left(\frac{1}{1-\rho},\frac{2\,\mu\,(e^{-\mu\,t}\,x)^{2\,(1-\rho)}}{\sigma^2\,(1-\rho)\,(1-e^{-2\,\mu\,(1-\rho)\,t})}\right)}\left(\frac{2\,\mu\,S_0^{2\,(1-\rho)}}{\sigma^2\,(1-\rho)\,\big(1-e^{-2\,\mu\,(1-\rho)\,t}\big)}\right).$$

  Notice that the formula is well-defined also for $\mu = 0$ as the respective limit for $\mu \searrow 0$ exists.

- There is a positive probability that the asset price diffuses to zero and remains there. However, there is also a positive probability that the asset price never touches zero. The probability that the asset price equals zero at time $t > 0$ is given by

$$\begin{aligned}\mathbb{P}(S_t = 0) &= \mathbb{P}\Big(T_{\delta_\rho} < \frac{\sigma^2\,(1-\rho)}{2\,\mu}\,\big(1-e^{-2\,\mu\,(1-\rho)\,t}\big)\Big)\\ &= 1 - F_{\chi^2\left(\frac{1}{1-\rho},0\right)}\left(\frac{2\,\mu\,S_0^{2\,(1-\rho)}}{\sigma^2\,(1-\rho)\,\big(1-e^{-2\,\mu\,(1-\rho)\,t}\big)}\right).\end{aligned}$$

  Notice that the formula is well-defined also for $\mu = 0$ as the respective limit for $\mu \searrow 0$ exists. Notice further that the limit as $t \to \infty$ of the last expression gives the probability that the asset price becomes zero at all, because the events $\{S_t = 0\}$ are decreasing in t (hitting zero once, the asset price remains there). Hence, there is also a positive probability that the stock price never touches zero, given by

$$\mathbb{P}(S_t > 0 \text{ for all } t > 0) = F_{\chi^2\left(\frac{1}{1-\rho},0\right)}\left(\frac{2\,\mu\,S_0^{2\,(1-\rho)}}{\sigma^2\,(1-\rho)}\right).$$

Because of the positive probability that the stock price process may diffuse to zero, the CEV model is used for a credit-equity pricing application in Atlan, Leblanc (2005). Moreover, Carr, Linetsky (2006) extend the derivation of Delbaen, Shirakawa (2002) by incorporating an additional jump-to-default component into the CEV stock price process, i.e. in addition to the fact that S may diffuse to zero, it is allowed that it suddenly jumps to default. The intensity of this sudden jump to default is defined in a reciprocal relationship with the asset price process in order to mimic the effect of a company's credit spread being a decreasing function of the same company's stock price. The findings of Carr, Linetsky (2006) are of a similar nature as the aforementioned result of Delbaen, Shirakawa (2002), in particular the non-central χ^2-distribution appears quite naturally.

3.2 The CIR model

A diffusion $Z = \{Z_t\}$ is called *CIR process* if it satisfies the stochastic differential equation

$$\mathrm{d}Z_t = (a - b\,Z_t)\,\mathrm{d}t + \sqrt{2\,\sigma\,Z_t}\,\mathrm{d}W_t,$$

for parameters $a, b \geq 0$, $\sigma > 0$. If $a \geq \sigma$, Z remains almost surely positive, while it may hit zero if this condition is violated. The CIR process obtains its name from Cox et al. (1985), who use it as a model for the stochastic evolution of interest rates. The idea is

to model the fictitious short rate process, from which the whole term structure of discount factors can be deduced, as a CIR process. The resulting model is one of the most tractable interest rate term structure models, hence it has become very popular. In particular, it is possible to evaluate in closed form the Laplace transform of the random variable $\int_0^t Z_s\, ds$ for $t > 0$, which provides closed formulas for discount factors within this model. The analytical availability of the latter Laplace transform is a property characterizing the family of so-called *additive processes*, of which the CIR process is the most prominent representative.
Now where does the non-central χ^2-distribution enter the scene? If $Y = \{Y_t\}$ is a squared Bessel process of dimension $2\,a/\sigma$, one can show that $Z = \{Z_t\}$ is a CIR process with parameters a, b, σ, where

$$Z_t := e^{-b\,t}\, Y_{\frac{\sigma}{2\,b}\left(e^{b\,t}-1\right)}, \quad t \geq 0.$$

Consequently, the importance of the non-central χ^2-law in the context of the CIR process stems from the fact that

$$\frac{2\,b\,Z_t}{\sigma\left(1-e^{-b\,t}\right)} \sim \chi^2\Big(\frac{2\,a}{\sigma}, \frac{2\,b\,Y_0}{\sigma\left(e^{b\,t}-1\right)}\Big), \quad t > 0.$$

The latter property may be used for the Maximum Likelihood estimation of the parameters of a CIR process, based on Maximum Likelihood estimation for the non-central χ^2-law. For instance, Kladivko (2007) provides a reader-friendly guide to a MATLAB implementation of the Maximum Likelihood method for the CIR process. Further applications of the CIR process in Mathematical Finance are as follows:

- It may be used as a model for the default intensity associated with the default time of a credit-risky asset, see, e.g., Duffie, Singleton (1999). The mathematical technique required for the derivation of survival probabilities within such a model is analogous to the technique involved in the classical CIR model for interest rate modeling: the Laplace transforms of the random variables $\int_0^t Z_s\, ds$, $t > 0$, basically yield the respective survival probabilities.

- Heston (1993) uses the CIR process in order to model the volatility of a stock price process as a stochastic process itself. Such stochastic volatility models have been introduced in order to explain the observed *volatility smile* present in stock option data. Like in the other aforementioned applications, the analytical tractability of the CIR process gives rise to the possibility to derive convenient numerical techniques for computing European stock option prices within the Heston model.

4 Conclusion

Key features of the non-central χ^2-distribution have been reviewed. It was pointed out how this probability law is related to the theory of squared Bessel processes, and why it appears in the context of Mathematical Finance.

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