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PUT = CDS + X

Jan-Frederik Mai
XAIA Investment GmbH
Sonnenstraße 19, 80331 München, Germany
jan-frederik.mai@xaia.com

Date: November 13, 2015

Abstract Long credit exposure in some company may be hedged either by buying CDS protection that triggers at a time point τ when a credit event with respect to the company is determined, or alternatively by buying put options on the company's stock price process $\{S_t\}_{t\geq 0}$ – at least when the assumption $\{\tau \leq T\} = \{S_T = 0\}$ is justified ("equity-jump-to-zero assumption"). Under this assumption, it is explained how the cost for put protection may be decomposed into two parts: (1) the cost for CDS protection (pure default protection) and (2) a remaining part that accounts for potential gains due to equity volatility (pure gamma). The proposed measurements are model-free, efficient to compute, and can help to get a feeling for how much default and gamma risk is priced into a specific put option.

1 The equity-jump-to-zero assumption

We consider a debt-issuing company whose stock price process is denoted by $\{S_t\}_{t\geq 0}$. In general, the stock price S_t at some future time point t is a non-negative random variable. Furthermore, we assume that the market trades credit default swaps (CDS) referencing on the company, and we denote by τ the unknown future time point at which a CDS credit event is triggered.

The present article is concerned with the hedging of a credit exposure in the considered company. Obviously, this can be done by buying CDS protection. Alternatively, since deteriorating credit risk is also likely to put pressure on the company's equity, it is also reasonable to think about hedging one's credit exposure by buying put options with underlying $\{S_t\}_{t\geq 0}$. While the presented thoughts are mostly model-free, there is one assumption we need to impose throughout in order to formally link the credit and equity components of the company: the so-called *equity-jump-to-zero assumption*, which reads as follows:

(A) For arbitrary $T > 0$ we assume that $\{S_T = 0\} = \{\tau \leq T\}$.

Intuitively, it means that the stock price is always strictly positive before τ, it jumps to zero at time τ, and then it stays at zero forever. The assumption reflects the idea that upon the arrival of τ the company is bankrupt, so the remaining value of the company is distributed among the debtors and no value remains for equity holders. Furthermore, we are not interested in the company anymore after τ, which is why it is sufficient to assume that $S_t = 0$ for all $t \geq \tau$.

Is assumption (A) realistic? Well, yes and no. In reality, a stock price process needs not jump to zero upon a credit event. It may still be positive for quite a while after τ, see Figure 1 for a recent example (Radioshack Corporation). However, a useful model is allowed to abstract from reality as long as it still captures the essential characteristics one is investigating. As we can also see from the Radioshack Corporation example in Figure 1, the stock price never recovered from its low level after τ, but stayed on a very low level for a couple of months before getting ultimately delisted (which corresponds to a jump to zero from the perspective of a put option holder). So on the one hand in this example the arrival of the credit event and the time when the stock price jumps to zero do not coincide. On the other hand, between these two time points nothing spectacular happened. From a practical perspective the assumption (A) may still be considered a convenient assumption because it makes a rigorous and useful mathematical model possible, without neglecting essential risks.

Fig. 1: Historical stock price process for Radioshack Corporation. The company was delisted in October 2015 at an ultimate stock price of about 2 cents per share. The red line in March 2015, when the stock traded still at about 15 cents per share, indicates the date when the ISDA Determinations Committee determined a CDS bankruptcy credit event. There is no sudden jump observable in the stock price process on the date of the CDS credit event. In contrast, the stock price process even traded slightly up after that for technical reasons, and then continuously lost value over time in a rather smooth fashion.

2 What are the costs for default protection?

Suppose we have a long exposure in some asset whose value is dominated by the credit risk of some company (e.g. a bond). We denote by N the amount we lose on this investment in case of an immediate default event (i.e. N is our current “jump-to-default exposure”). We seek to install a hedging position that eliminates our jump-to-default exposure. We have two ideas on the table which do this job for us: (1) buying CDS protection referencing on the company or (2) buying put options on the stock of the underlying company. How can we compare these two possibilities?

2.1 CDS protection costs

We denote the time of default by τ, and by $\mathbb{P}$ a risk-neutral pricing measure. Given that, if we buy CDS protection for maturity T with annualized running CDS spread c, the market price (i.e. the CDS upfront) for the respective protection leg is given by

$$CDS(T) := (1-R)\,\mathbb{E}[DF(\tau)\,1_{\{\tau\leq T\}}] - c\int_0^T DF(t)\,\mathbb{P}(\tau>t)\,\mathrm{d}t,$$

where $R\in[0,1]$ denotes the recovery rate (which is assumed to be known and constant). We seek the CDS nominal N_C such that in case of an immediate default event the PnL of our CDS amounts to

$$N_C\,(1-R-CDS(T)) \stackrel{!}{=} N, \text{ i.e. } N_C = \frac{N}{1-R-CDS(T)}.$$

This implies that our total CDS hedge costs for jump-to-default-neutrality amount to $N_C\cdot CDS(T)$, which equals

$$\frac{N\cdot CDS(T)}{1-R-CDS(T)}.$$

We denote the last expression, divided by the nominal N, by CCPUN(c,R), i.e.

$$\text{CCPUN}(c,R) := \frac{CDS(T)}{1-R-CDS(T)}.$$

The acronym "CCPUN" stands for CDS costs per unit nominal.

2.2 Put protection costs

The PnL of an American put option with maturity T and strike K in case of an immediate default amounts to $K-P_T(K)$, denoting by $P_T(K)$ the price of the put. Consequently, we need $N/(K-P_T(K))$ put options in order to hedge our exposure of N. These cost us an amount of

$$N\cdot\text{PCPUN}(K) := N\,\frac{P_T(K)}{K-P_T(K)},$$

where the acronym "PCPUN" is introduced in an analogous manner as for CDS in the previous paragraph and stands for put costs per unit nominal.

2.3 Decomposition of put protection costs

Let us neglect mark-to-market risk, and only consider CDS protection and put protection from a cash flow perspective. Whereas our CDS hedge only pays off at τ if $\tau\leq T$ (otherwise there are only cash outflows), the put may also be exercised before τ with positive PnL (even though no default takes place). This is illustrated in Figure 2, which visualizes the payoff profile of a put. Assuming absence of arbitrage, the put protection until maturity T must necessarily be more expensive than the respective CDS protection[1] until T. In particular, we observe that

$$\begin{aligned}\text{CCPUN}(c,R) \leq \text{CCPUN}(0,R) &= \frac{\mathbb{E}[DF(\tau)\,1_{\{\tau\leq T\}}]}{1-\mathbb{E}[DF(\tau)\,1_{\{\tau\leq T\}}]}\\ \leq \text{PCPUN}(K) &= \frac{P_T(K)/K}{1-P_T(K)/K}.\end{aligned}$$

[1] If this is not the case, then this is either a striking arbitrage opportunity or a strong indication that assumption (A) does not hold.

It is important to notice that the "all-upfront CDS protection cost" CCPUN$(0, R)$ = CCPUN(0) is independent of R. As $K \searrow 0$, in non-pathological situations we have that[2]

$$\frac{P_T(K)}{K} \longrightarrow \frac{\partial}{\partial K} P_T(0) = \mathbb{E}[DF(\tau)\, 1_{\{\tau \le T\}}], \tag{1}$$

so that the put hedge costs PCPUN(K) converge to the maximal CDS hedge costs CCPUN(0), which are paid when the whole insurance is paid upfront (zero CDS running spread). Consequently, the value CCPUN(0) separates the cheaper CDS protection costs (which depend on the choice of c) and the more expensive put protection costs (which depend on the choice of K).

Remark 2.1 (European put)
If the put option was European-style, then (1) would change to

$$\frac{P_T(K)}{K} \longrightarrow \frac{\partial}{\partial K} P_T(0) = \mathbb{E}[DF(T)\, 1_{\{\tau \le T\}}],$$

which follows immediately from the bounded convergence theorem. Since in the current low interest rate environment we have $DF(\tau) \approx DF(T)$ on $\{\tau \le T\}$ for short maturities T, for most practical purposes the difference between American-style and European-style put options is only of secondary importance.

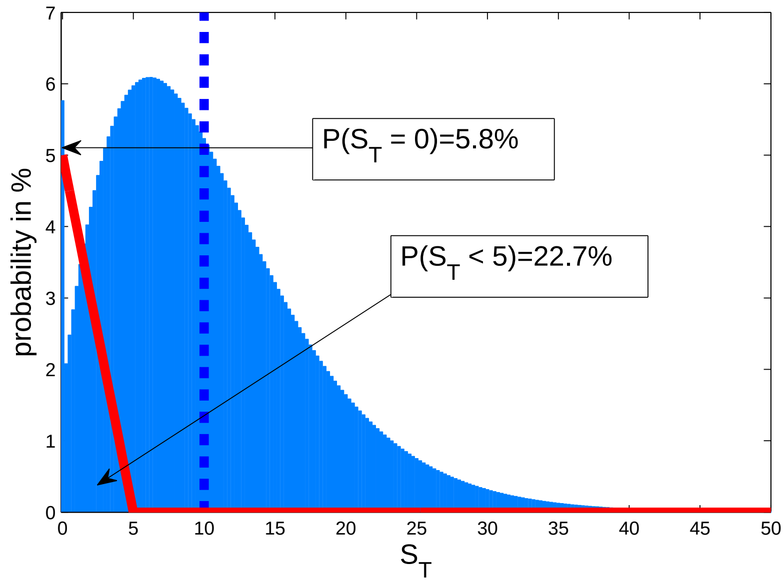

Fig. 2: Histogram of the probability distribution of S_T (in light blue). The dotted dark blue line shows the expected value $\mathbb{E}[S_T] = 10$. The red line visualizes the payoff profile of a put option with strike 5. While the CDS only triggers with probability $\mathbb{P}(S_T = 0)$, the put option has a non-zero payoff with probability $\mathbb{P}(S_T < 5)$.

These considerations imply that it is reasonable to define

$$D_T(K) := \frac{\text{CCPUN}(0)}{\text{PCPUN}(K)} = \frac{\mathbb{E}[DF(\tau)\, 1_{\{\tau \le T\}}]\,(K - P_T(K))}{P_T(K)\,\big(1 - \mathbb{E}[DF(\tau)\, 1_{\{\tau \le T\}}]\big)},$$

$$M_T(K) := 1 - D_T(K) \in [0, 1].$$

[2] See the Appendix for an explanation.

We may interpret $D_T(K)$ as the proportion of the put protection costs that are associated to pure default risk protection, and $M_T(K)$ as the proportion of the put protection costs that are associated with pure gamma risk protection. By definition, D_T (M_T) is decreasing (increasing) in the strike price. The numbers $D_T(K)$ and $M_T(K)$ are model-free measurements that can help to clarify how much default and gamma risk is priced into a specific put option. In order to compute them the only thing which is desired is the distribution function of τ (i.e. default probabilities), which may easily be extracted from an observed CDS curve.
While obviously both CDS costs and put costs increase in case a company becomes more distressed, it is a priori unclear how the proportions D_T and M_T behave. However, Figure 3 demonstrates the functions $D_T(K)$ and $M_T(K)$ for three real-world cases with different levels of creditworthiness – all with similar maturities. The depicted numbers have been computed within a defaultable Markov diffusion model[3] that has been calibrated jointly and successfully to CDS and put prices. Judging from these three examples, it appears as if not only the total default protection costs CCPUN(0) increase with decreasing creditworthiness (which is clearly expected), but also the relative share of default risk protection $D_T(K)$ within the observed put prices increases for at-the-money strike levels. For far out-of-the-money strike prices ($\leq 20\%$ moneyness), however, the examples show that $D_T(K)$ is not a monotone measure in the creditworthiness in general. This indicates that the value of $D_T(K)$ (or equivalently $M_T(K)$) is highly case-sensitive – which in turn makes a measurement of $D_T(K)$ interesting.

3 Conclusion

Under the assumption that the stock price jumps to zero in case of a CDS credit event it was shown how to compare default risk hedge costs via put options and default risk hedge costs via CDS protection. We demonstrated how to split the put hedge costs into two components, one accounting for pure default risk and the other accounting for gamma risk. Based on this decomposition, we proposed a model-free measurement for how much default risk protection is implicitly contained in a put option price.

Appendix: Justification of (1)

Denoting by $\mathcal{T}_{[0,T]}$ the set of all stopping times (w.r.t. market filtration, which is the natural filtration of $\{S_t\}_{t\geq 0}$) with values in $[0,T]$,

$$\frac{P_T(K)}{K} = \sup_{\eta \in \mathcal{T}_{[0,T]}} \mathbb{E}\Big[DF(\eta)\Big(1-\frac{S_\eta}{K}\Big)_+\Big].$$

We prove the claimed equality by establishing the lower and upper bounds separately. The lower bound is obvious:

$$\begin{aligned}
&\sup_{\eta \in \mathcal{T}_{[0,T]}} \mathbb{E}\Big[DF(\eta)\Big(1-\frac{S_\eta}{K}\Big)_+\Big] \\
&\qquad \geq \mathbb{E}\Big[DF(\tau)\,1_{\{\tau\leq T\}}\Big] + \mathbb{E}\Big[DF(T)\Big(1-\frac{S_T}{K}\Big)_+ 1_{\{\tau>T\}}\Big] \\
&\qquad \geq \mathbb{E}\Big[DF(\tau)\,1_{\{\tau\leq T\}}\Big],
\end{aligned}$$

[3] More precisely, the JDCEV model of Carr, Linetsky (2006).

where the first inequality is obtained by considering the particular stopping time $\eta = \min\{\tau, T\}$. In order to derive the more difficult estimate from above, we need to impose the following assumption:

(B) In case of a certain default before τ (i.e. conditioned on the event $\{\tau \leq T\}$), it is optimal to exercise the put option at τ.

Remark 3.1 (Intuition of assumption (B))
Assumption (B) says that the magnitude of a potential interest rate loss induced by exercising the option at some time t, for which $DF(t)$ is rather low, is negligible compared with the magnitude of a potential gain induced by the ultimate jump to default $\Delta S_\tau := S_{\tau-} - S_\tau$ of the stock price. Within typical pricing models, like defaultable Markov diffusion models, this is an acceptable assumption since the value ΔS_τ often takes a non-negligible positive value, whereas the range of the function $t \mapsto DF(t)$ on $[0, T]$ is rather narrow for typical option maturities T (which are rather short, say $T \leq 2$). In practice, however, it might happen that the equity market anticipates the default time τ earlier than the actual date on which the ISDA Determinations Committee determines the CDS credit event, cf. Figure 1. On the one hand, this provides an incentive to exercise one's put option earlier than τ and invest the received cash at the risk-free rate. On the other hand, in times of low or even negative (risk-free) interest rates such considerations are at best of secondary importance.

We proceed with the justification, taking assumption (B) for granted. Let $n \in \mathbb{N}$ arbitrary. The distribution function F of the random variable $\inf_{t\in[0,T]} S_t$ satisfies $F(0+) := \lim_{K\downarrow 0} F(K) = F(0)$ (since any distribution function has left limits everywhere). Consequently, there exists a $K_n > 0$ such that

$$\mathbb{P}\big(\inf_{t\in[0,T]} S_t \in (0, K_n)\big) < \frac{1}{n \sup_{t\in[0,T]} DF(t)}.$$

We define the following two disjoint events:

$$A_n := \Big\{\inf_{t\in[0,T]} S_t \in (0, K_n)\Big\}, \quad B_n := \Big\{\inf_{t\in[0,T]} S_t \geq K_n\Big\}.$$

Notice in particular that $A_n \cup B_n = \{\tau > T\}$, so that $A_n \cup B_n \cup \{\tau \leq T\}$ defines a (disjoint) partition of Ω. Furthermore, on the event B_n obviously $DF(\eta)\left(1 - \frac{S_\eta}{K_n}\right)_+ = 0$ for all $\eta \in \mathcal{T}_{[0,T]}$. Consequently,

$$\begin{aligned}
&\mathbb{E}\Big[DF(\eta)\left(1 - \frac{S_\eta}{K_n}\right)_+\Big] \\
&\quad = \mathbb{E}\Big[1_{\{\tau\leq T\}}\underbrace{DF(\eta)\left(1 - \frac{S_\eta}{K_n}\right)_+}_{\leq DF(\tau)\text{ by (B)}}\Big] + \mathbb{E}\Big[1_{A_n} DF(\eta)\underbrace{\left(1 - \frac{S_\eta}{K_n}\right)_+}_{\leq 1}\Big] \\
&\quad \leq \mathbb{E}\Big[DF(\tau)\,1_{\{\tau\leq T\}}\Big] + \mathbb{P}(A_n)\sup_{t\in[0,T]} DF(t) \\
&\quad \leq \mathbb{E}\Big[DF(\tau)\,1_{\{\tau\leq T\}}\Big] + \frac{1}{n}.
\end{aligned}$$

The first inequality above relies on assumption (B). Since the right-hand side is independent of η, we conclude that

$$\sup_{\eta\in\mathcal{T}_{[0,T]}} \mathbb{E}\Big[DF(\eta)\Big(1-\frac{S_\eta}{K_n}\Big)_+\Big] \le \mathbb{E}\Big[DF(\tau)\,1_{\{\tau\le T\}}\Big] + \frac{1}{n}.$$

We may without loss of generality arrange[4] that $K_n \to 0$ as $n \to \infty$, so that we have found one particular null sequence of strike prices (K_n) satisfying $\lim_{n\to\infty} P(K_n)/K_n = \mathbb{E}\big[DF(\tau)\,1_{\{\tau\le T\}}\big]$. Assuming differentiability[5] of the function $K \mapsto P(K)$ at zero then yields the claim, because it implies that $\lim_{n\to} P(K_n)/K_n$ is invariant with respect to the choice of null sequence (K_n).

References

P. Carr, V. Linetsky, A jump to default extended CEV model: an application of Bessel processes, *Finance and Stochastics* **10** (2006), pp. 303–330.

[4] E.g. via replacement of (K_n) by $(\tilde{K}_n)$, where $\tilde{K}_n := \min\{K_n, 1/n\}$.

[5] From a practical perspective, this assumption is not severe because (a) due to a lack of observations an assumption on differentiability is required anyway and assuming non-differentiability at zero would be rather unnatural, and (b) in typical pricing models empirical observations imply differentiability.

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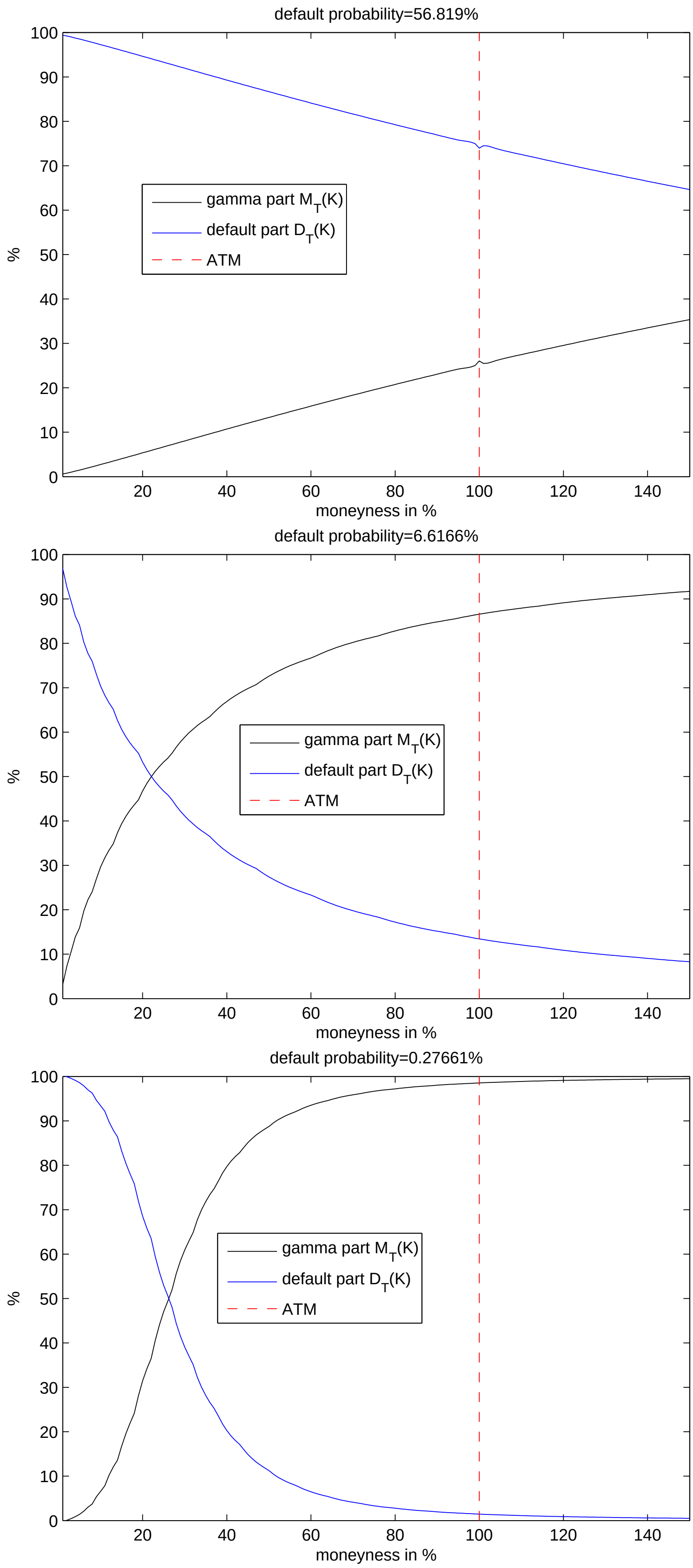

Fig. 3: Decomposition of put protection costs into D_T and M_T for three names of different creditworthiness ($T \approx 1.25$ years for each name). Top: highly distressed ($\mathbb{P}(\tau \leq T) = 58.8\%$), Middle: moderately distressed ($\mathbb{P}(\tau \leq T) = 6.6\%$), Bottom: not distressed ($\mathbb{P}(\tau \leq T) = 0.3\%$).