XAIA INVESTMENT

THE INFORMATION CONTENT OF STOCK OPTION DATA

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Abstract We consider market bid and ask prices for European call options on some stock with a fixed maturity and for different strike levels. It is well–known that these options provide information about the (risk-neutral) probability distribution of the future stock price. But how much? In this note we aim at explaining how rich the information of the given option data is.

1 Introduction

We observe a set of European call option prices on a stock, with the same maturity $T > 0$ and for a battery of strike prices $0 \leq K_0 < K_1 < \ldots < K_n < \infty$. The respective market prices are denoted by $\underline{C}_0, \underline{C}_1, \ldots, \underline{C}_n$ (bid prices) and $\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_n$ (ask prices), respectively, and satisfy $0 \leq \underline{C}_i \leq \overline{C}_i$ for all $i = 0, \ldots, n$. The observed bid-ask spread corresponds to transaction costs we face in case we buy or sell the options. We denote the (non-negative) stock price at maturity T by S_T and the price of a risk-free zero coupon bond with maturity T by $DF(0,T)$. A typical task of a financial analyst is to find a solution to the following problem:

Problem 1.1 (Calibration problem)
Find a probability measure $\mathbb{Q}$ so that we have the inequalities $\underline{C}_i \leq C_i \leq \overline{C}_i$ for all $i = 0, \ldots, n$, where

$$C_i := DF(0,T)\, \mathbb{E}^{\mathbb{Q}}[\max\{S_T - K_i, 0\}]. \quad (1)$$

Without loss of generality we may assume $DF(0,T) = 1$ for the remainder of this article, since we can simply rewrite Problem 1.1 in terms of the prices $\tilde{\underline{C}}_i := \underline{C}_i / DF(0,T)$ and $\tilde{\overline{C}}_i := \overline{C}_i / DF(0,T)$, respectively. Further, the measure $\mathbb{Q}$ typically must satisfy an additional constraint, namely $\mathbb{E}^{\mathbb{Q}}[S_T] = F(0,T)$ for a given forward equity price $F(0,T)$. However, this case is included in Problem 1.1 by setting $K_0 := 0$ and $C_0 := F(0,T)$. Classical arbitrage pricing theory implies that the found call option prices C_i are arbitrage–free in the following sense: under perfect market conditions[1] there exists no portfolio in the risk-free zero bond, the stock, and the call options, which yields a risk-free return above the return of the available risk-free zero coupon bond. By the reverse logic, a solution $\mathbb{Q}$ to Problem 1.1

[1] Short-selling is allowed, no transaction costs (in particular, the options can be bought and sold at prices C_i, i.e. no bid-ask), etc..

may be used to infer what the market thinks about the full probability distribution of S_T, provided we believe in an arbitrage-free market which approximately satisfies the perfect market conditions. In other words, the quoted option prices relinguish partial information about a probability distribution of S_T, which is called a risk-neutral probability distribution. Any risk-neutral probability distribution must be viewed as an extrapolation of this partial information to full information about the probability distribution of S_T – and this extrapolation is by no means unique in general, as will be illustrated in the sequel.

2 When does a solution exist at all?

Assume for a minute that there is a solution $\mathbb{Q}$ to Problem 1.1. In this case, it follows immediately from $K_{i-1} \leq K_i$ and the representation (1) that

$$C_{i-1} \geq C_i, \quad i = 1, \ldots, n, \tag{2}$$

i.e. the sequence C_i of mid call prices is non-increasing. It is furthermore observed that

$$\begin{aligned} C_{i-1} - C_i &= \mathbb{E}^{\mathbb{Q}}[\max\{S_T - K_{i-1}, 0\} - \max\{S_T - K_i, 0\}] \\ &\leq \mathbb{E}^{\mathbb{Q}}[S_T - K_{i-1} - (S_T - K_i)] = K_i - K_{i-1}, \quad i = 1, \ldots, n, \end{aligned} \tag{3}$$

which implies that the slope of the decrease is bounded from below by $-1 \leq (C_i - C_{i-1})/(K_i - K_{i-1})$. Moreover, the function $K \mapsto \mathbb{E}^{\mathbb{Q}}[\max\{S_T - K, 0\}]$ is obviously convex. In particular, this implies that

$$-1 \leq \frac{C_i - C_{i-1}}{K_i - K_{i-1}} \leq \frac{C_{i+1} - C_i}{K_{i+1} - K_i}, \quad i = 1, \ldots, n-1, \tag{4}$$

where the lower bound of -1 on the left hand side follows from (3). The following lemma states that if a sequence of call prices C_i is non-increasing with slope greater than -1 and convex, i.e. satisfies (2) and (4), then there exists always a measure $\mathbb{Q}$ such that the representation (1) is valid.

Lemma 2.1 (A trivial risk-neutral measure)
If $C_0, C_1, \ldots, C_n$ is a sequence of non-negative numbers satisfying (2) and (4), then there exists a probability measure $\mathbb{Q}$ such that the C_i can be represented as in (1).

Proof
Recall our assumption $DF(0, T) = 1$. Without loss of generality we assume that $K_0 = 0$. (If not, we simply extend the given sequence. More precisely, we let $K_{-1} = 0$ and set C_{-1} to any value between $C_0 + K_0\,(C_0 - C_1)/(K_1 - K_0)$ and $C_0 + K_0$. Notice that this is possible, since $(C_1 - C_0)/(K_1 - K_0) \geq -1$ by assumption. Then we shift the whole sequence by one index up, so that we have a sequence indexed by $i = 0, \ldots, n+1$, which fits into the hypothesis of the lemma.) We additionally set $C_{n+1} := 0$ and $K_{n+1} = \infty$. With this definition, we have extended (2) and (4) to

$$\begin{aligned} C_i &\geq C_{i+1}, \quad i = 0, \ldots, n, \\ -1 \leq \frac{C_i - C_{i-1}}{K_i - K_{i-1}} &\leq \frac{C_{i+1} - C_i}{K_{i+1} - K_i}, \quad i = 1, \ldots, n. \end{aligned} \tag{5}$$

Therefore, we are able to define the numbers $D_0 := 1$, $D_{n+1} := 0$, and

$$D_i := -\Big(\alpha_i \, \frac{C_{i+1} - C_i}{K_{i+1} - K_i} + (1 - \alpha_i) \, \frac{C_i - C_{i-1}}{K_i - K_{i-1}}\Big), \quad i = 1, \dots, n, \tag{6}$$

for arbitrary numbers $\alpha_i \in (0, 1)$. It is an immediate consquence of the second line in (5) that $1 = D_0 \geq D_1 \geq \dots \geq D_n \geq D_{n+1} = 0$. Further, we define for $i = 0, \dots, n$ the sequence

$$\bar{K}_i := \begin{cases} \frac{C_i + K_i \, D_i - (C_{i+1} + K_{i+1} \, D_{i+1})}{D_i - D_{i+1}} & \text{, if } D_i > D_{i+1} \\ K_i & \text{, else} \end{cases}.$$

It is important to notice that

$$\bar{K}_i \, (D_i - D_{i+1}) = C_i + K_i \, D_i - (C_{i+1} + K_{i+1} \, D_{i+1}),$$

even for those i with $D_i = D_{i+1}$ (because in this case the last equation reads $0 = 0$, as can be checked easily). It is again a consequence of (5) that $K_i \leq \bar{K}_i \leq K_{i+1}$ for all $i = 0, \dots, n$. To see this, we observe by immediate manipulations that

$$K_i \leq \bar{K}_i \leq K_{i+1} \;\Leftrightarrow\; D_i \geq -\frac{C_{i+1} - C_i}{K_{i+1} - K_i} \geq D_{i+1},$$

which is true by (5) and the definition of the D_i. Now we can define the measure $\mathbb{Q}$ such that S_T takes on only finitely many different values, namely

$$\mathbb{Q}(S_T = \bar{K}_i) = D_i - D_{i+1} \geq 0, \quad i = 0, \dots, n.$$

Obviously, $\mathbb{Q}$ is a probability measure, since $D_0 - D_{n+1} = 1$. We observe further by a telescope sum argument that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\max\{S_T - K_i, 0\}] &= \sum_{j=i}^{n} (\bar{K}_j - K_i) \, (D_j - D_{j+1}) \\ &= \sum_{j=i}^{n} (C_j + K_j \, D_j) - (C_{j+1} + K_{j+1} \, D_{j+1}) \\ &\qquad - K_i \sum_{i=j}^{n} (D_j - D_{j+1}) \\ &= C_i + K_i \, D_i - K_i \, D_i = C_i. \end{aligned}$$

Hence, $\mathbb{Q}$ is such that the given numbers C_i can be represented as in (1). $\square$

The numbers D_i in the proof of Lemma 2.1 can be interpreted as digital prices. More precisely, under the constructed solution $\mathbb{Q}$ it holds true that

$$D_i = \mathbb{Q}(S_T > K_i), \quad i = 0, \dots, n+1.$$

Consequently, the choice of the D_i in (6) specifies how much probability mass the measure $\mathbb{Q}$ assigns to each strike bucket. By virtue of Lemma 2.1 Problem 1.1 may be reformulated in a simpler form.

Problem 2.2 (Calibration problem, reformulated)
Find a sequence $C_0, \dots, C_n$ of non-negative numbers satisfying (2) and (4), such that $\underline{C}_i \leq C_i \leq \overline{C}_i$ for all $i = 0, \dots, n$.

If we have found a solution $C_0, C_1, \dots, C_n$ to Problem 2.2, the proof of Lemma 2.1 immediately provides us with a solution to Problem 1.1. Problem 2.2 can easily be solved within fractions of a second by means of linear programming. More precisely, we propose to solve the following two linear programs:

$$\left\{\begin{array}{l}\text{minimize } \sum_{i=0}^n C_i \\ \text{subject to: } \underline{C}_i \leq C_i \leq \overline{C}_i,\ i = 0, \dots, n,\ A\,\vec{C} \leq \vec{b}\end{array}\right\} \quad \text{(LP1)}$$

$$\left\{\begin{array}{l}\text{maximize } \sum_{i=0}^n C_i \\ \text{subject to: } \underline{C}_i \leq C_i \leq \overline{C}_i,\ i = 0, \dots, n,\ A\,\vec{C} \leq \vec{b}\end{array}\right\} \quad \text{(LP2)}$$

The matrix[2] $A = (A_{i,j}) \in \mathbb{R}^{2\,n \times (n+1)}$ and the vector $\vec{b} = (b_i) \in \mathbb{R}^{2\,n}$ represent the conditions (2) and (4). The non-zero entries of A and $\vec{b}$ are given as follows (all other entries are zero):

$$A_{i,j} = \begin{cases} -1 & ,\ 1 \leq i = j+1 \leq n \\ 1 & ,\ 1 \leq i = j \leq n \end{cases},$$

$$A_{n+i,j} = \begin{cases} \frac{-1}{K_i - K_{i-1}} & ,\ 1 \leq i = j+1 \leq n-1 \\ \frac{K_{i+1} - K_{i-1}}{(K_i - K_{i-1})\,(K_{i+1} - K_i)} & ,\ 1 \leq i = j \leq n-1 \\ \frac{-1}{K_{i+1} - K_i} & ,\ 1 \leq i = j-1 \leq n-1 \end{cases},$$

$$A_{2\,n,j} = \begin{cases} 1 & ,\ j = 0 \\ -1 & ,\ j = 1 \end{cases}, \qquad b_{2\,n} = K_1 - K_0.$$

The idea of the linear program (LP1) (resp. (LP2)) is to find the admissible mid quotes C_i that are closest to the bid quotes $\underline{C}_i$ (resp. ask quotes $\overline{C}_i$). They can be solved by the standard Simplex algorithm. For example, this is pre-implemented in MATLAB as the routine `linprog`. Both linear programs (LP1) and (LP2) return a solution to Problem 2.2 (if existent at all), say $\vec{C}_1$ and $\vec{C}_2$. And any convex combination $\vec{C} := \alpha\,\vec{C}_1 + (1-\alpha)\,\vec{C}_2$, for arbitrary $\alpha \in [0, 1]$, yields an admissible solution to Problem 2.2. Figure 1 visualizes these solutions in a real-world example.

3 Solutions with an additional interpretation

The proof of Lemma 2.1 is constructive. It can be applied to any solution $C_0, \dots, C_n$ of Problem 2.2 in order to yield a solution $\mathbb{Q}$ to Problem 1.1. However, this construction only provides one particular solution, which is by no means unique. Already from the proof of Lemma 2.1 we observe that we are free to choose the $\alpha_i \in (0, 1)$ in (6), each choice potentially leading to a different solution. Furthermore, all the constructed solutions specify $\mathbb{Q}$ in such a way that the stock price S_T can only take on at most $n+1$ different values. As mentioned earlier, the probability measure $\mathbb{Q}$

[2] Notice that the column index j runs through $j = 0, \dots, n$.

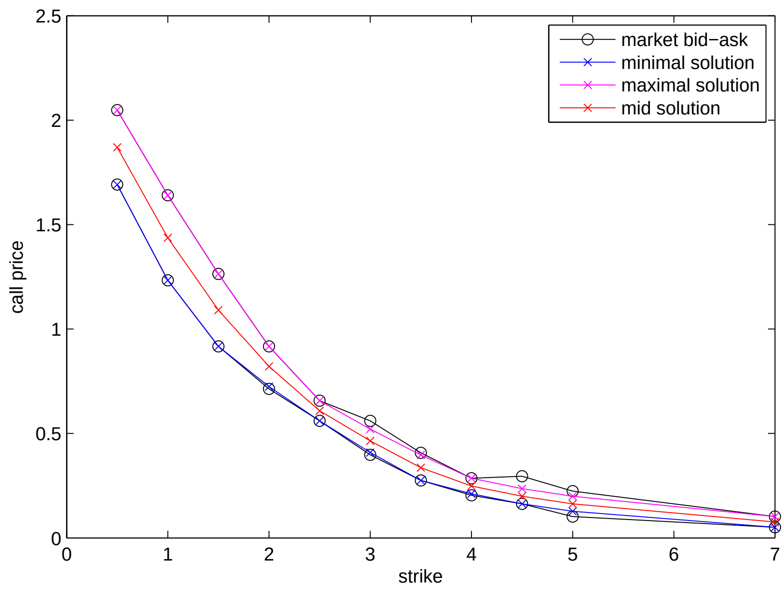

Fig. 1: Market-observed bid and call strikes, together with three solutions to Problem 2.2. One is the maximal solution of (LP2), one the minimal solution of (LP1), and the third is the mean of the first two.

basically serves as a tool to infer information about the (market's opinion about the) distribution of S_T. Depending on the specific application, it might be appropriate to postulate additionally that a solution $\mathbb{Q}$ to Problem 1.1 is such that the implied probability law of S_T satisfies additional properties. This results in a more difficult problem in general.

For instance, it is often desired that the probability distribution of S_T under $\mathbb{Q}$ is absolutely continuous, i.e. there exists a non-negative function $f : [0,\infty) \to [0,\infty)$ such that $\mathbb{Q}(a \leq S_T \leq b) = \int_a^b f(x)\,\mathrm{d}x$ for $0 \leq a \leq b \leq \infty$. The function f is called the *density* of S_T. In this case, it follows that the function

$$C(K) := \mathbb{E}^{\mathbb{Q}}[\max\{S_T - K, 0\}] = \int_K^\infty (x-K)\,f(x)\,\mathrm{d}x$$

is twice differentiable with $C^{''}(K) = f(K)$ for all $K > 0$. Similar to Lemma 2.1, one can show the following.

Lemma 3.1 (A risk-neutral measure with positive density)
If $C_0, C_1, \ldots, C_n$ is a sequence of non-negative numbers satisfying (2) and (4) with strict inequalities, then there exists a probability measure $\mathbb{Q}$ such that the C_i can be represented as in (1) and such that S_T admits a strictly positive density under $\mathbb{Q}$.

Proof
In the proof of Lemma 2.1 we choose constants D_i in (6) from the interval

$$\Big[-\frac{C_{i+1}-C_i}{K_{i+1}-K_i}, -\frac{C_i - C_{i-1}}{K_i - K_{i-1}}\Big],$$

for $i = 1, \ldots, n$. If we have strict inequalities in (4), then none of these intervals is a singleton, so it is possible to choose all the

D_i from the interior of the corresponding intervals. Under this condition, the results of Neri, Schneider (2012) imply that there exists a piecewise exponential (in particular positive) density f on $(0, \infty)$ such that

$$C_i = \int_{K_i}^{\infty} (x - K_i)\, f(x)\, \mathrm{d}x, \quad D_i = \int_{K_i}^{\infty} f(x)\, \mathrm{d}x,$$

for all $i = 0, \ldots, n$. □

For the convenience of the reader, the Appendix briefly outlines another, even slightly simpler and numerically more robust, density construction than the one based on Neri, Schneider (2012) that was mentioned in the proof of Lemma 3.1. The density $f = f_D$ in the proof of Lemma 3.1 is piecewise exponential and depends on the chosen values of the D_i. It is shown in Neri, Schneider (2012) to maximize entropy among all densities explaining the given call (C_i) and digital (D_i) prices. Dropping the digital prices D_i, it is furthermore shown in Neri, Schneider (2013) that there exists a unique density f which is piecewise exponential and continuous, and explains the given call prices C_i. The latter density is of the form $f = f_D$ for some particular D (i.e. is also piecewise exponential) and is called the *Buchen-Kelly density*, referring to pioneering work by Buchen, Kelly (1996). Intuitively, maximizing entropy when determining the measure $\mathbb{Q}$ (respectively when finding the density f_D or f) means that one only uses the given option data, but uses no other economic reasoning[3]. On the one hand, maximizing entropy is an elegant way to reduce model risk. On the other hand, the resulting densities may be very unsmooth. In the next section we present examples of such densities, based on the call option data in Figure 1.
The typical situation in practice is that the distribution of S_T under a solution $\mathbb{Q}$ to Problem 1.1 is postulated to stem from a parametric family of probability distributions. In mathematical terms, one demands that under the measure $\mathbb{Q}$ it holds that $S_T \sim F_\theta$, where $\{F_\theta\}_{\theta \in \Theta}$ is a family of distribution functions, parameterized by parameters $\theta \in \mathbb{R}^m$. This situation happens, for example, if the stock price process is modeled under $\mathbb{Q}$ as a stochastic process from some parametric family. A typical example for a parametric law of S_T is a mixture of lognormals, see, e.g., Brigo, Mercurio (2002). In such a parametric setup the calibration problem boils down to finding parameters θ such that

$$\underline{C}_i \leq C_i(\theta) := \int_{K_i}^{\infty} (x - K_i)\, \mathrm{d}F_\theta(x) \leq \overline{C}_i, \tag{7}$$

for all $i = 0, \ldots, n$. This will almost always be a highly non-linear problem, and it is typically solved by minimizing a non-negative error functional Err(θ) saytisfying Err$(\theta) = 0$ if and only if θ satisfies (7), e.g.

$$\text{Err}(\theta) := \sum_{i=0}^{n} \max\{C_i(\theta) - \overline{C}_i, 0\} + \max\{\underline{C}_i - C_i(\theta), 0\}.$$

[3] Entropy-maximization also relies on (restrictive?) assumptions. For example, the Buchen-Kelly density is always light-tailed so that all moments of S_T exist. In other words, heavy tails never maximize entropy.

If the obtained minimum equals zero, a solution to Problem 1.1 of the desired parametric form exists. If the minimum is greater than zero, it depends on the application one has in mind whether or not the minimizing probability law $\mathbb{Q} = \mathbb{Q}(\theta)$ is an acceptable approximation or not.

4 How different are possible solutions?

We provide an example of how different possible solutions to Problem 1.1 can look. To this end, we reconsider the call option data of Figure 1, which stem from a real-world example. The strikes of the observed call options (with a maturity in $T = 1.715$ years) span a range from 0.5 to 7 USD. With the current stock price being equal to 2.31 USD, this means that a wide range from 21.65% to 303.03% moneyness is covered. This is a rather unusual situation, but it means that the given call option data should provide quite a lot of information about the market's opinion about the probability distribution of S_T. In other words, the given data specify a solution to Problem 1.1 quite accurately in the sense that any two solutions $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are expected to be quite similar, i.e. $\mathbb{Q}_1 \approx \mathbb{Q}_2$. Let us check if this is indeed the case.

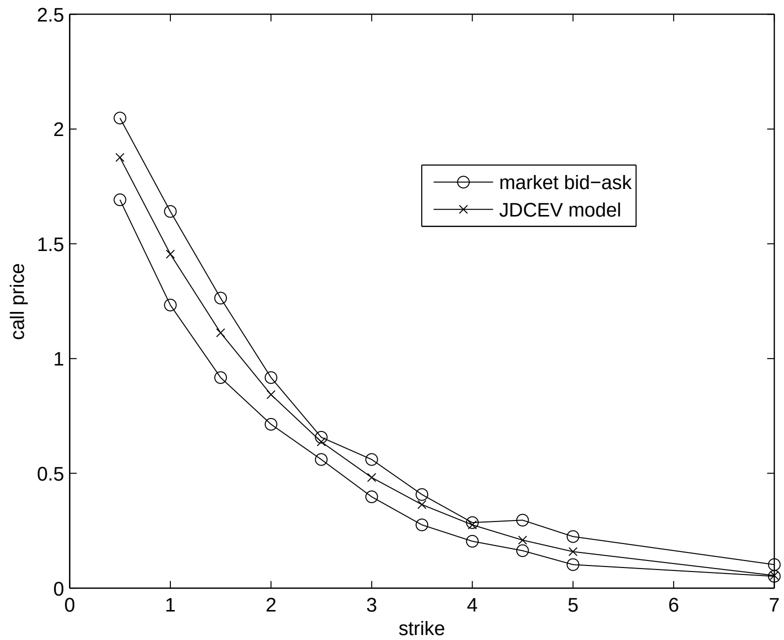

Fig. 2: Visualization of call option prices resulting from a JDCEV model that is fitted so that it provides a solution to Problem 1.1.

First of all, we fit a so-called JDCEV model[4] of Carr, Linetsky (2006) to the given call option data. As Figure 2 shows, this is possible and the model-implied call option prices C_i provide proper mid quotes. It is interesting to mention that this model implies a solution $\mathbb{Q}$ to Problem 1.1 satisfying $\mathbb{Q}(S_T = 0) > 0$, so the law of S_T is not absolutely continuous in this case (but not discrete either). Second, we carry out the construction from the proof of Lemma 2.1 with the model prices C_i as input. We end up with a discrete probability distribution for S_T. Concerning the involved

[4] JDCEV stands for jump-to-default extended constant elasticity of variance. The JDCEV model is a popular credit-equity model.

choice of the D_i in (6), we apply the convention $\alpha_i = 0.5$ for each $i = 0, \ldots, 10$. Third, we take the same D_i (and the C_i) as input for the algorithm presented in Neri, Schneider (2012), which yields a maximum entropy density (MED) for S_T. This density is piecewise exponential and discontinuous. Finally, by optimizing the chosen constants D_i using the optimization procedure described in Neri, Schneider (2013) we compute the Buchen-Kelly density for S_T, which also explains all given call prices C_i. The latter is the maximum entropy density when only the prices C_i (but not the D_i) are used as specifying information. All four resulting distribution functions for S_T are visualized in Figure 3.

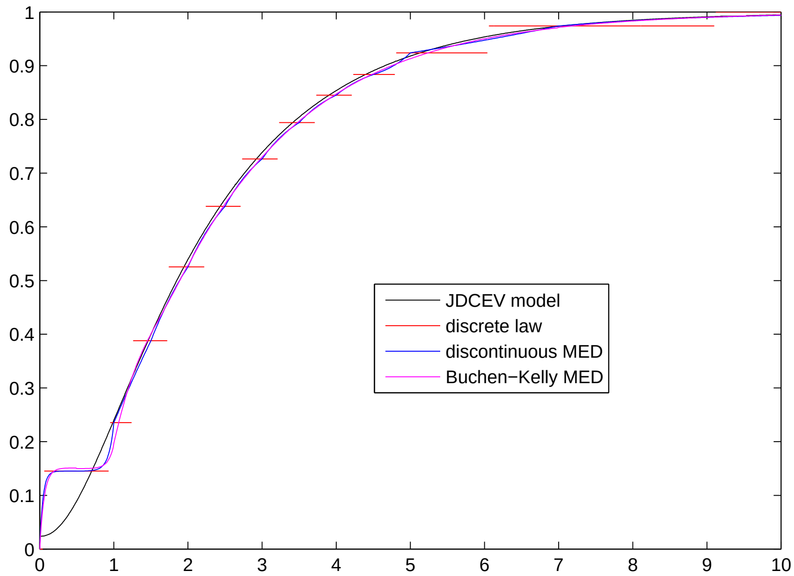

Fig. 3: Visualization of distribution functions for S_T under different solutions to Problem 1.1. All four distribution functions imply precisely the same call option prices.

It is observed that the JDCEV model implies an atom at zero, i.e. the respective distribution function (black line in Figure 3) has a jump at zero (to a level of about 0.0225). For all other three distribution functions zero is a fix point. The distribution function associated with the discrete probability law (red line in Figure 3) is a step function jumping at all possible values S_T can take. This is clearly the distribution function that sticks out most of the four functions. Indeed, the distribution functions associated with the JDCEV model and the two distribution functions arising from entropy maximization appear to be very similar on the interval $[1, 10]$. If one looks closely, one can observe that the distribution function belonging to the Buchen-Kelly MED (magenta line in Figure 3) is less bumpy than the one associated with the discontinuous MED (blue line in Figure 3). On the interval $[0, 1]$, however, the JDCEV model is quite different from the other distribution functions. The reason is that the JDCEV model assigns point mass to zero, whereas the other distribution functions do not. Intuitively, the other three distribution functions “catch up” this point mass backlog very quickly, explaining their steeper increase after zero compared with the JDCEV model. The observation that the four models differ most on the interval $[0, 0.5]$ is also intuitive when taking into account the fact that this interval lies outside the range spanned by the strikes of the given option data. This corresponds to a range for which information must be extrapolated under a significant amount of freedom concerning the distributional assumptions.

As a final remark, let us notice that in a situation where the strikes of the observed call option data only span a range of, say, 80% to 120% moneyness, which is not unusual in the marketplace, the dissimilarity of possible solutions to Problem 1.1 can be huge. This is simply due to the fact that there is a lot of freedom for the probability distribution of S_T outside the specified moneyness range. Figure 4 illustrates this by means of a fictitious example. First, using the Black–Scholes formula[5], we compute call option prices for strike levels ranging from 80% to 120% moneyness. Second, using the construction from the proof of Lemma 2.1, we find a discrete probability law yielding the same strike prices. Third, we run the algorithm of Neri, Schneider (2012) to construct a (discontinuous) maximum entropy density yielding also the same strike prices. Finally, we run the algorithm of Neri, Schneider (2013) to construct a continuous maximum entropy density, the so-called Buchen-Kelly density, also yielding the same strike prices. It is observed that all four constructed distribution functions differ massively outside the moneyness range that is spanned by the observed options, whereas within this range they are all very similar.

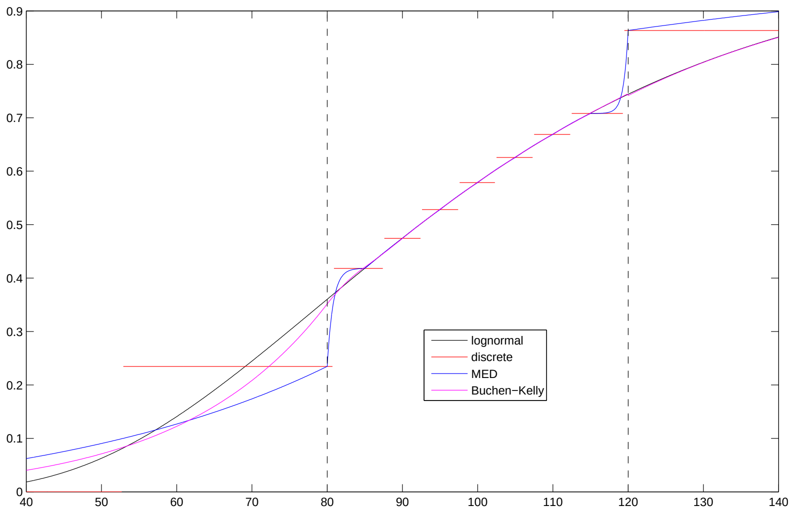

Fig. 4: Visualization of four different distribution functions for S_T, which all imply the same call option prices within a strike range of 80% to 120% moneyness (this range is indicated by the dotted vertical lines).

There is also another way to explain this ambiguity. Figure 5 visualizes the densities corresponding to the three absolutely continuous risk-neutral probability measures depicted in Figure 4. It is observed that the discontinuous MED assigns disproportionately much mass to the marginal strike buckets $[80, 85]$ and

[5]With current stock price $S_0 = 100$, volatility $\sigma = 0.4$, interest rate $r = 0$, and time to maturity $T = 1$.

$[115, 120]$. Referring to the proof of Lemma 2.1, these probability masses are controlled by the choice of the digital prices D_i in (6). In contrast, the continuous Buchen-Kelly MED has been derived via the algorithm in Neri, Schneider (2013) which optimizes the D_i in such a way that a continuous MED is obtained. The potential range we may choose each D_i from in the general case of the algorithm in Neri, Schneider (2012) is the widest (i.e. the ambiguity the largest) for the marginal strike buckets, with the consequences being depicted in Figures 4 and 5. In real-world applications, it might hence make sense to at least impose modeling assumptions for the values D_1 and D_n, which determine the probability mass outside the observed moneyness range. In the particular example here, it is possible to assign between 8% to 70% of the probability mass outside the observed strike range, without losing the ability to explain the given call prices. Clearly, how much probability mass we think lies outside the observed strike range has a strong effect.

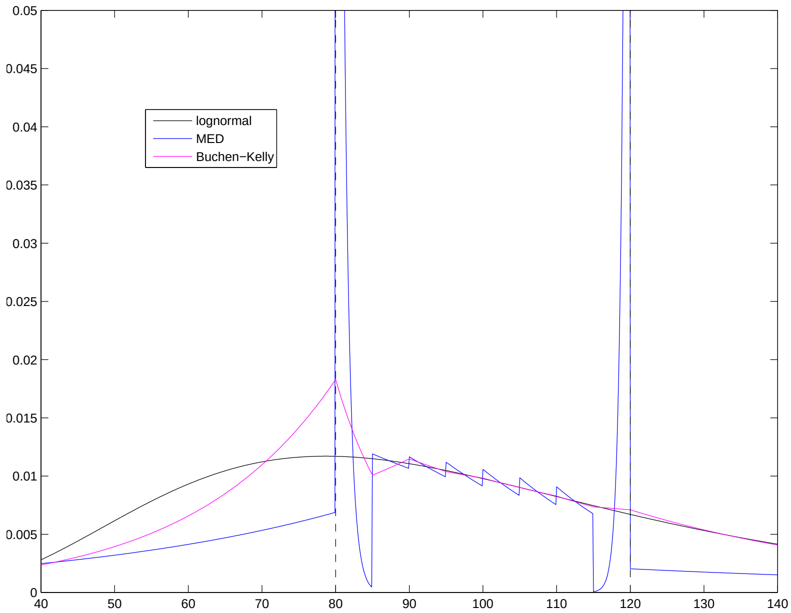

Fig. 5: Visualization of different risk-neutral densities, all yielding the same call option prices.

5 Conclusion

The goal of this article was to convey a feeling for how much information one can retrieve from observed stock option data. To this end, it was first explained that a battery of European call option strikes – if it satisfies certain monotonicity and convexity conditions – can always be represented as expectation values with respect to a discrete risk-neutral measure. It was highlighted how diverse the set of such representing risk-neutral measures can be.

Appendix: Piecewise linear cdf

Although the algorithm of Neri, Schneider (2012), which has been applied before, always returns a density for given call and digital option prices, the algorithm sometimes runs into numerical problems. This is because the piecewise exponentiality of the density is prone to extreme spikes near observed strike prices,

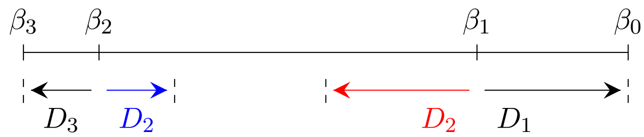

Fig. 6: Since the intervals $[\beta_3, \beta_2]$ and $[\beta_1, \beta_0]$ are much smaller than the middle interval $[\beta_2, \beta_1]$, it is impossible to choose D_2 in such a way that (8) holds for all i. The colored arrows indicate possible choices for D_2 which do not violate (8) for $i = 1$ (red) and $i = 2$ (blue), but these two ranges have an empty intersection.

as can be observed, e.g., in Figure 5. In order to obtain a more stable algorithm, which always returns a numerically convenient risk-neutral density, one may replace the piecewise exponential form by a piecewise constant one, at least within the strike range $[K_1, K_n]$. More precisely, we briefly show how to change the construction in Lemma 2.1 in order to obtain a distribution function for S_T which is continuous on $[K_1, K_n]$ (piecewise linear instead of piecewise constant). We are able to define a density f for S_T under $\mathbb{Q}$ on each of the given strike intervals constant as

$$f\big|_{(K_i,K_{i+1}]}(x) = \begin{cases} \frac{(D_i - D_{i+1})\, 1_{(2\,\bar{K}_i - K_{i+1}, K_{i+1})}(x)}{2\,(K_{i+1} - \bar{K}_i)} & \text{, if } \bar{K}_i > \frac{K_i + K_{i+1}}{2} \\ \frac{(D_i - D_{i+1})\, 1_{(K_i, 2\,\bar{K}_i - K_i)}(x)}{2\,(\bar{K}_i - K_i)} & \text{, else} \end{cases},$$

for $i = 1, \ldots, n-1$. Obviously, this definition then implies for $i = 1, \ldots, n-1$ that

$$\int_{K_i}^{K_{i+1}} f(x)\,\mathrm{d}x = D_i - D_{i+1},$$
$$\frac{1}{D_i - D_{i+1}} \int_{K_i}^{K_{i+1}} f(x)\,x\,\mathrm{d}x = \bar{K}_i.$$

Obviously, this density is piecewise constant (hence the corresponding distribution function piecewise linear and continuous). However, the density has a hole on the interval $[K_i, K_{i+1}]$ if and only if $\bar{K}_i$ is not precisely the midpoint of the interval. This raises the question whether it is possible to choose the D_i in such a way that the $\bar{K}_i$ equal the midpoints of the intervals, in order to eliminate holes. For the sake of a simplified notation we define

$$\beta_i := -\frac{C_{i+1} - C_i}{K_{i+1} - K_i}, \quad i = 0, \ldots, n.$$

For $i = 1, \ldots, n-1$, simple algebraic manipulations imply that

$$\bar{K}_i = \frac{K_i + K_{i+1}}{2} \Leftrightarrow \frac{D_i + D_{i+1}}{2} = -\frac{C_{i+1} - C_i}{K_{i+1} - K_i} = \beta_i. \tag{8}$$

Condition (8) shows that its is not always possible to eliminate holes, because we might encounter a situation where it is impossible to place D_{i-1} and D_i symmetrically around β_{i-1}, and simultaneously place D_i and D_{i+1} symmetrically around β_i. Figure 6 illustrates this problem for $i = 2$.

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