# How to COMPUTE AN <br> ANNUALIZED INCOME FOR A BOND FROM ITS Z-SPREAD? 

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#### Abstract

The so-called Z-spread of a bond is often used as a "quick-anddirty" approximation for the bond's annualized income. However, it is pointed out that the Z -spread is earned on the bond's market value rather than on the bond's nominal, which can make a crucial difference for bond's trading far away from par. Concerning implications, such an annualized income computation is an integral part of the negative basis measurement according to the so-called Z-spread methodology, which should be adjusted according to the suggestions of the present note.


1 The proxy formula We start by introducing the required notations.
$z$ : bond's Z-spread
$B$ : bond's current clean market value (per unit notional)
$N$ : bond's nominal
Recall that the $Z$-spread of a bond is a measure for its excess return relative to some reference interest rate curve. More precisely, we denote by $D F(0, t)$ today's value of a zero coupon bond with maturity $t$, so that every cash flow at time $t$ needs to be multiplied with $D F(0, t)$ in order to compute its net present value. The discount factors $D F(0, t)$ are typically interpreted as "risk-free", which means that they are derived from standard interest rates bearing minimal credit risk. Corporate bonds typically bear more credit risk, which means that the sum of all their discounted cash flows is higher than the bond's market value. In order to force the value of a bond being equal to the sum of its discounted cash flows it is common to introduce one additional parameter into the discount factors. Indeed, a cash flow at time $t$ is multiplied with $D F(0, t) \exp (-z t)$ for some real number $z$. The value $z$ which is required in order to explain the bond's market value as the sum of its cash flows, discounted in this way, is called its Z-spread. If $z>0$, this intuitively means that the present value of the bond is below the present value of an equivalent risk-free bond. Furthermore, the value $z$ is sometimes interpreted as the excess return over the risk-free rate which the bond pays in order to compensate for its credit risk. As a consequence, it is common to compute the bond's expected annualized income as $z N$. However, we point out below that this is not always a good idea. First, let us make clear that we define the expected annualized income of the bond as the difference between the expected bond's value in one year, discounted back into today by multiplication of $D F(0,1)$, and today's value $B$. In particular, this
implies that a risk-free zero coupon bond with maturity one year has expected annualized income equal to zero. The following proxy formula should be used for computing the bond's expected annualized income, given no default and no changing market conditions during one year:

$$
\begin{equation*}
\text { expected annualized income } \approx\left(e^{z}-1\right) B N \text {. } \tag{1}
\end{equation*}
$$

The approximative nature of Formula (1) has two sources: (a) the computation of the bond's value change is based on the Zspread philosophy, which is just one possible "accrual model", and (b) the precise timing of the coupon payments within the next year is abstracted from in order to obtain a simplified formula. Let us collect a couple of remarks regarding Formula (1) before providing a justification in the next section.
(a) Why Z-spread on market value and not on nominal?

We provide an example showing that the approximation $z N$, which is sometimes applied by market participants, can be completely wrong, see also the example in the last section. Consider a zero coupon bond with maturity one year, a current price of $B=0.5$, and unit nominal $N=1$. The readers will agree that the expected annualized income should equal $D F(0,1)-B$, because the bond accrues up to one within the next year, if no default occurs. Indeed, Formula (1) yields precisely this result, because the Z -spread is defined by the equation $e^{-z} D F(0,1)=B$ in this case, implying $e^{z} B=D F(0,1)$. However, the alternative formula $z N$ may yield the completely wrong result, depending on the risk-free rate. Assume for example that the one-year risk-free zero rate equals $1 \%$ so that $D F(0,1)=\exp (-0.01) \approx 0.99005$, then

$$
z N=\log \left(\frac{D F(0,1)}{B}\right) N=\log (1.9801) \approx 0.683147,
$$

overestimating the desired value $0.99005-0.5=0.49005$.
(b) What about the further approximation $e^{z}-1 \approx z$ ?

If one has no computer at hand, it might be tempting to approximate the involved exponential function by its first-order term, i.e. $e^{z}-1 \approx z$. This implies the simplified formula
annualized income $\approx z B N$.
However, be aware that this induces a systematic bias, because $e^{z}-1 \geq z$ and strictly " $>$ " whenever $z>0$, which is the usual case.
(c) Implications for negative basis measurement: The articles Bernhart, Mai (2012); Mai (2014) deal with the appropriate measurement of the negative basis between a CDS and an eligible bond. One of the best-known measurement approaches is the so-called Z -spread method. The latter computes the negative basis as the difference between annualized income, measured in terms of a Z -spread, and the annualized costs, measured in terms of a CDS spread. However,
the present note implies that the further is earned on the basis package's market value, whereas the latter is paid on the CDS nominal. The present note points out that the Z-spread might not always be an appropriate measurement for the annualized income of the basis package, see (a) above. Formula (1) shows how the Z -spread method for negative basis measurement should be revised in order to be more accurate: when $\alpha$ denotes the fraction of CDS nominal divided by bond nominal, and when $z$ denotes the $Z$-spread on the basis package ${ }^{1}$, then the negative basis - measured in terms of bond nominal - should be computed according to the formula

$$
N B_{\alpha}^{(Z *)}:=\left(e^{z}-1\right)(B+\alpha u p f)-\alpha s,
$$

where upf and $s$ denote the upfront and running coupon of the respective CDS. Depending on how far the basis package trades away from par, this formula might differ significantly from the traditional formula $N B_{\alpha}^{(Z *)}=z-\alpha s$, cf. (Bernhart, Mai, 2012, Algorithm 2(v)).
(d) Warning: Be aware that Formula (1) assumes that the bond is plain vanilla, i.e. no sinking fund, no coupon step-up, no call rights, etc.. However, generalizations of the Z-spread concept to bonds with odd coupons and sinking fund features are straightforward. Even generalizations to call and optional sinking fund features are possible, but more involved, cf. Mai (2013).

2 Justification of Formula (1) Let us introduce some more notation in order to be able to write down a formal proof. We introduce a time argument $t$ and denote the bond's market price at time $t$ by $B(t)$. Further, we denote by $D F(t, T)$ a discount factor at time $t$ for the later time point $T>t$. Finally, denote the bond's coupon/ redemption payment dates by $t_{1}, \ldots, t_{n}$, and the bond's clean cash flow ${ }^{2}$ at time $t_{k}$ by $C_{k}(t)$. The bond's Z -spread $z$ is defined by the equation

$$
B(0)=\sum_{k: t_{k}>0} C_{k}(0) D F\left(0, t_{k}\right) e^{-z t_{k}}
$$

Furthermore, under the assumption of no changing market conditions (i.e. no default, $Z$-spread remains constant), the bond price until time $t$ will accrue up or down to the value

$$
B(t)=\sum_{k: t_{k}>t} C_{k}(t) D F\left(t, t_{k}\right) e^{-z\left(t_{k}-t\right)} .
$$

We observe

$$
\begin{aligned}
B(1) & D F(0,1)=e^{z} \sum_{k: t_{k}>1} C_{k}(1) D F\left(0, t_{k}\right) e^{-z t_{k}} \\
& =e^{z}\left(B(0)-C \sum_{\substack{k: t_{k}>0 \\
t_{k-1} \leq 1}} D F\left(0, t_{k}\right) e^{-z t_{k}} \times\right. \\
& \left.\times\left(\left(1-t_{k-1}\right) 1_{\left\{t_{k}>1\right\}}+\left(t_{k}-\max \left\{t_{k-1}, 0\right\}\right) 1_{\left\{t_{k} \leq 1\right\}}\right)\right) .
\end{aligned}
$$

[^0]Now let's have a closer look at the remaining sum. We denote the number of equidistant coupon payments per year by $n$ (so that "annual" means $n=1$, "semi-annual" means $n=2$, and so on), and assume the last coupon payment before 0 was at $-\epsilon$, where $0 \leq \epsilon<1 / n$. This means that the coupon payment dates $t_{k}$ satisfying $t_{k}>0$ and $t_{k-1} \leq 1$ are of the form $1 / n-\epsilon, \ldots,(n+$ $1) / n-\epsilon$. Hence, the remaining sum can be written as

$$
\begin{aligned}
& S_{\epsilon, n}(z):=\frac{1}{n} \sum_{k=1}^{n} e^{-z(k / n-\epsilon)} D F\left(0, \frac{k}{n}-\epsilon\right) \\
&-\epsilon e^{-z\left(\frac{1}{n}-\epsilon\right)}\left(D F\left(0, \frac{1}{n}-\epsilon\right)-e^{-z} D F\left(0, \frac{n+1}{n}-\epsilon\right)\right) .
\end{aligned}
$$

Under the assumption of an annual coupon payment frequency (meaning $n=1$ ), and the assumption that the last coupon has just been paid at $t=0$ while the next coupon is paid precisely in one year at $t=1$ (meaning $\epsilon=0$ ), this sum equals precisely $S_{0,1}(z)=e^{-z} D F(0,1)$.
Now let us assume we have at time $t=0$ a portfolio consisting of only the bond, i.e. our portfolio is worth $B(0) N$ at $t=0$. The value of our portfolio at time $t=1$, discounted back into $t=0$ in order to be compared with $B(0) N$, consists of the bond's discounted market value $D F(0,1) B(1) N$ as well as of all collected, discounted coupon payments in $[0,1]$, given precisely by the value $S_{\epsilon, n}(0) C N$. Summing up, this means that our total income in one year equals

$$
\begin{aligned}
&(D F\left.(0,1) B(1)+S_{\epsilon, n}(0) C-B(0)\right) N \\
& \quad=\left(\left(e^{z}-1\right) B(0)+C\left(S_{\epsilon, n}(0)-e^{z} S_{\epsilon, n}(z)\right)\right) N \\
& \quad \approx\left(e^{z}-1\right) B(0) N
\end{aligned}
$$

where the last approximation $e^{z} S_{\epsilon, n}(z) \approx S_{\epsilon, n}(0)$ is exact for $n=1$ and $\epsilon=0$, but constitutes a bias in general. The size of this bias is visualized in Figure 1. It is observed that the error is increasing in the number of coupon payments per year.

3 A small, numeric example Inspired by a real-world example, we consider a bond with maturity December 2020, $B=0.58$ (bond way below par), $C=$ $9.875 \%$ (high coupon rate), yielding a high Z-spread of $z=2070$ bps (computed from the Bloomberg screen YAS). We furthermore assume unit nominal $N=1$, yielding

$$
\left(e^{z}-1\right) B N=\left(e^{0.207}-1\right) 0.58 \approx 0.13339
$$

In contrast, the (wrong) formula $z N$ would yield 0.2070 , which is obviously too high. The first-order approximation $z B N \approx 0.12006$ is too low.

4 Conclusion It was pointed out that the product of the Z-spread with the nominal is in general not an accurate approximation of a bond's annualized income. It was pointed out that it is more accurate to multiply this value additionally with the bond's market value. Furthermore, it was shown that the Z-spread $z$ in this computation serves only as the first-order approximation of the more accurate factor $e^{z}-1$, which is to be preferred over $z$ - and not much harder to compute.


Fig. 1: Visualization of $z \mapsto e^{z} S_{\epsilon, n}(z) \approx S_{\epsilon, n}(0)$ for different $n$, $\epsilon=0$ (top) and $\epsilon=0.2$ (bottom). The discount factors were assumed to equal $D F(t, T)=\exp (-0.01(T-t))$

References G. Bernhart, J.-F. Mai, Negative basis measurement: finding the holy scale, XAIA homepage article (2012).
J.-F. Mai, Z-spreads for bonds with optional sinking feature: a Bellman exercise, XAIA homepage article (2013).
J.-F. Mai, Ein zweiter Blick auf die negative Basis, XAIA homepage article (2014).


[^0]:    ${ }^{1}$ This is defined as the Z-spread on the bond, when the bond's market price is assumed to equal $B+\alpha u p f$, cf. (Bernhart, Mai, 2012, Algorithm 2).
    ${ }^{2}$ The clean cash flows depend on $t$ and are given by $C_{k}(t)=C\left(\max \left\{t_{k}, t\right\}-\right.$ $\left.\max \left\{t_{k-1}, t\right\}\right)$ for all $k<n$ and $C_{n}(t)=1+C\left(t_{n}-\max \left\{t_{n-1}, t\right\}\right)$.

