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Can we retrieve the equity forward from American option prices?

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Abstract The equity forward $F(0,T)$ with maturity T is defined as today's risk-neutral expectation of a stock price at time T. If arbitrage-free prices $P(K)$ and $C(K)$ for European put and call options with maturity T are observed for different strikes K, the equity forward can be retrieved from the put-call parity. In particular, $F(0,T)$ is invariant with respect to different risk-neutral pricing measures which explain observed option prices, i.e. it is a model-free quantity. More precisely, $F(0,T)$ is given by the unique root of the (in practice partially) observed function $K \mapsto C(K) - P(K)$. If only American put and call option prices are observed, the lack of a put-call parity makes it more difficult to retrieve $F(0,T)$ in an unambiguous way from the observed option data. In particular, the unique root of $K \mapsto C(K) - P(K)$ in general is no longer equal to the equity forward. The present article investigates whether American put and call prices also determine the quantity $F(0,T)$ unambiguously. Unfortunately, this seemingly simple "yes or no"-question appears to be non-trivial and open, and the present investigation is not able to answer it.

1 Introduction

Throughout this article we denote a stock price process by $S = \{S_t\}_{t\geq 0}$ and a deterministic, risk-free, continuous discounting rate by $r(.)$. Furthermore, we denote by $\delta(.) \geq 0$ a continuous rate accounting for proceeds from stock possession, either through dividends or through stock lending. Within this setup, arbitrage pricing theory suggests that for any risk-neutral pricing measure $\mathbb{Q}$ we have that

$$F(0,T) := \mathbb{E}_{\mathbb{Q}}[S_T] = S_0\, e^{\int_0^T r(t)-\delta(t)\,\mathrm{d}t}, \quad T \geq 0. \tag{1}$$

In other words, the so-called *equity forward* with maturity T, denoted $F(0,T)$, is independent of $\mathbb{Q}$, i.e. does not depend on the stochastic behavior of the stock, except for its drift rate $r(.)-\delta(.)$. While S_0 is observable and $r(.)$ is bootstrapped from interest rate sensitive derivatives, the rate $\delta(.)$ - and hence the equity forward $F(0,T)$ - is a priori an unobservable quantity. However, it becomes observable when European put and call options with different maturites are observed, as will be explained in the sequel.
We denote by $C(K)$ and $P(K)$ the observed market prices for a call option and a put option with maturity T and strike K on the stock S. Assuming that these prices are arbitrage-free and the options are European-style, there exists a (not necessarily unique) risk-neutral pricing measure $\mathbb{Q}$ such that

$$C(K) = e^{-\int_0^T r(t)\,\mathrm{d}t}\,\mathbb{E}_{\mathbb{Q}}[(S_T-K)_+],$$
$$P(K) = e^{-\int_0^T r(t)\,\mathrm{d}t}\,\mathbb{E}_{\mathbb{Q}}[(K-S_T)_+]. \tag{2}$$

From these representations it is straightforward to derive the so-called *put-call parity*, which reads

$$C(K) - P(K) = e^{-\int_0^T r(t)\,\mathrm{d}t}\big(F(0,T) - K\big). \tag{3}$$

The function $f(K) := C(K) - P(K)$ is strictly decreasing with unique root $F(0,T)$ by (3). In practice, the function f is observed on a grid $K_1 < K_2 < \ldots < K_n$ which is reasonably fine around the root $F(0,T)$, so that a linear interpolation of observed values provides a satisfying approximation of f, hence of $F(0,T)$. Consequently, $F(0,T)$ is essentially an observed quantity and the rate $\delta(.)$ may be read off from (1) under knowledge of S_0, $F(0,T)$, and $r(.)$. For instance, if European options are observed for several maturities $T_1 < T_2 < \ldots < T_m$, then $\delta(.)$ may be specified in a piecewise constant manner so that (1) is satisfied for each $T \in \{T_1, \ldots, T_m\}$.
Now what if the observed prices $C(K)$ and $P(K)$ correspond to American-style options, which is the usual case in practice? In this case, the put-call parity (3) needs not hold and the argument above to read off $F(0,T)$, respectively $\delta(.)$, from observed quantities breaks down. In particular, the root of the function f is no longer given by $F(0,T)$ in general.
The remainder of the article is organized as follows. Section 2 reviews prominent literature dealing with the pricing of American-style options, while Section 3 explains why it is difficult to learn about $\delta(.)$ when only observing American-style option prices.

2 American options

A nice survey of different perspectives on American-style options is found in Broadie, Detemple (2004), from which we briefly recall general representations. Because of the right to exercise before maturity, the American-style call and put option prices may be written as suprema[1] over stopping times as

$$C(K) = \operatorname*{ess\,sup}_{\eta \in \mathcal{T}[0,T]} \Big\{ \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_0^\eta r(t)\,\mathrm{d}t}\,(S_\eta - K)_+ \Big] \Big\},$$
$$P(K) = \operatorname*{ess\,sup}_{\eta \in \mathcal{T}[0,T]} \Big\{ \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_0^\eta r(t)\,\mathrm{d}t}\,(K - S_\eta)_+ \Big] \Big\},$$

where $\mathcal{T}[0,T]$ denotes the set of all stopping times with respect to the market filtration taking values in $[0,T]$. These formulas are very intuitive, in particular when compared to their European-style counterparts (2). There exists a second representation for American-style calls and puts, the so-called *early exercise premium representation*, which decomposes them into a sum of their European-style counterpart and an early exercise premium. Since it is educational, we carry out the derivation for the put in the sequel. Denoting by τ_p the optimal stopping time for the put option, we make the following assumption:

(AP) There is a function $E_p : [0,T] \to [0,\infty]$ such that

$$\tau_p = \inf\{t > 0 \,:\, S_t \le E_p(t)\}.$$

[1] The supremum of measurable functions in general needs not be measurable, e.g. $f := 1_A$ for a non-measurable set A equals the supremum of the functions $f_B := 1_B$, $B \subset A$ measurable (easy to check). This fact requires the technical notion of the essential supremum in the call and put formulas.

The function E_p is called *exercise boundary* for the put option. In words, this assumption means that it is optimal to exercise the put option if the stock price falls below the (ex ante unknown) exercise boundary. Lemma 1 in the Appendix of Detemple, Tian (2002) shows that Assumption (AP) is satisfied in a large family of diffusion models and Jacka (1991) shows that is satisfied in the Black-Scholes model. However, in a general, model-free context we need to formally make Assumption (AP) without verification in order to proceed[2]. Furthermore, the following argument requires the assumption that the stock price trajectories $t \mapsto S_t$ can have no upward jumps. Given this and Assumption (AP), and denoting the market price of the put option with strike K at time $t \in [0, T]$ by $P_t(K)$, consider the following trading strategy, cf. Carr et al. (1992):

(0) Buy one American put option at $t = 0$ for the price $P_0(K)$ (assuming $S_0 > E_p(0)$).

(1) At the first time t_1 when $S_{t_1} < E_p(t_1)$, the put is exercised and we receive its intrinsic value $K - S_{t_1}$. With this money, we can finance to put the amount K into the risk-free bank account and short-sell one stock.

(2) At the next time $t_2 > t_1$ when $S_{t_2} \geq E_p(t_2)$, we liquidate our existing position and buy one American put option for the amount $P_{t_2}(K) \geq K - S_{t_2}$. Under the assumption that $t \mapsto S_t$ does not exhibit upward jumps, it holds that $P_{t_2}(K) = K - S_{t_2}$. Furthermore, our current portfolio wealth is

$$K\, e^{\int_{t_1}^{t_2} r(s)\,\mathrm{d}s} - S_{t_2} - \int_{t_1}^{t_2} \delta(t)\, S_t\, e^{\int_t^{t_2} r(s)\,\mathrm{d}s}\,\mathrm{d}t,$$

where the last summand corresponds to dividend earnings we miss due to our shortselling investment. Hence, the portfolio rebalancing (i.e. buy the put and liquidate existing portfolio) costs us precisely

$$K - S_{t_2} - \Big(K\, e^{\int_{t_1}^{t_2} r(s)\,\mathrm{d}s} - S_{t_2} - \int_{t_1}^{t_2} \delta(t)\, S_t\, e^{\int_t^{t_2} r(s)\,\mathrm{d}s}\,\mathrm{d}t\Big)$$
$$= \int_{t_1}^{t_2} \big(\delta(t)\, S_t - K\, r(t)\big)\, e^{\int_t^{t_2} r(s)\,\mathrm{d}s}\,\mathrm{d}t.$$

(3) We wait for the next time point t_3 at which the stock crosses the exercise boundary from above and proceed like in (1), then we wait for the next crossing from below and proceed like in (2), and so on.

The wealth process W_t of this strategy at some arbitrary time point $t \in [0, T]$ is obviously given by

$$W_t = \max\{P_t(K), K - S_t\} - \int_0^t \big(\delta(s)\, S_s - K\, r(s)\big)\, e^{\int_s^t r(u)\,\mathrm{d}u}\, 1_{\{S_s \leq E_p(s)\}}\,\mathrm{d}s.$$

[2] A separation of the real line into two connected parts corresponding to "exercise" or "continuation" appears natural on first glimpse. However, how to make sure that a separation into three connected parts corresponding to "exercise", "continuation", and "exercise if an independent coin toss yields heads" is suboptimal? Think of Neyman Pearson theory, in which randomized tests are optimal in a certain sense!

The discounted portfolio wealth process needs to be a martingale under $\mathbb{Q}$, so that

$$P_0(K) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_0^T r(t)\,\mathrm{d}t}\, W_T\Big] = e^{-\int_0^T r(t)\,\mathrm{d}t}\,\mathbb{E}_{\mathbb{Q}}\big[(K-S_T)_+\big] + \mathbb{E}_{\mathbb{Q}}\Big[\int_0^T \big(K\,r(s)-\delta(s)\,S_s\big)\,e^{-\int_0^s r(u)\,\mathrm{d}u}\,1_{\{S_s\leq E_p(s)\}}\,\mathrm{d}s\Big].$$

Consequently, the American-style put price is decomposed into the sum of a European-style put price and an *early exercise premium*. Since we know that the American-style put price is greater or equal to the European-style put price, the early exercise premium term must be non-negative. In particular, we can learn from this fact a trivial bound for the exercise boundary, namely $E_p(t) \leq \max\{K\,r(t)/\delta(t), 0\}$, which is an intuitive bound that becomes infinity (hence trivial) for positive $r(t)$ and $\delta(t) \downarrow 0$. In particular, we observe that the American put option is not exercised at a time point t for which $r(t) \leq 0$.
The analogous derivation for the call option relies on the following assumption:

(AC) There is a function $E_c : [0,T] \to [0,\infty]$ such that

$$\tau_c = \inf\{t > 0 \,:\, S_t \geq E_c(t)\}.$$

Under Assumption (AC), the same logic as for the put option implies

$$C(K) = e^{-\int_0^T r(t)\,\mathrm{d}t}\,\mathbb{E}_{\mathbb{Q}}\big[(S_T-K)_+\big] + \mathbb{E}_{\mathbb{Q}}\Big[\int_0^T \big(\delta(s)\,S_s-K\,r(s)\big)\,e^{-\int_0^s r(u)\,\mathrm{d}u}\,1_{\{S_s\geq E_c(s)\}}\,\mathrm{d}s\Big],$$

where $E_c : [0,T] \to [0,\infty]$ denotes the exercise boundary for the call option. The call exercise boundary satisfies $E_c(t) \geq K\,r(t)/\delta(t)$, which shows that the call is never exercised if $\delta(.) \equiv 0$, which is a well-known fact.
As the more involved representations of American-style options suggest, implying the probability distribution of the underlying stock price from observed option prices is much more difficult than in the case of European-style options. In theory, if the function $K \mapsto C(K)$ of European call options is observed, so is the probability law of S_T. In particular, if $K \mapsto C(K)$ is twice differentiable, the second derivative equals the density of S_T. In particular, the expectation of S_T, i.e. the forward $F(0,T)$, is observable. This fact is no longer true if $C(K)$ corresponds to American-style call options, as can easily be seen by the following example.

Example 2.1 ($K \mapsto C(K)$ does not determine $F(0,T)$)
Assume that $r \leq 0$ and $\delta \geq 0$ are constant, and $\mathrm{d}S_t = S_t\,(r-\delta)\,\mathrm{d}t$ behaves deterministically. Then $C(K) = (S_0-K)_+$, and there is no chance to observe δ, or the expected value of S_T, only from the function $K \mapsto C(K) = (S_0-K)_+$ - intuitively because the time T never occurs in the rationale of optimal option execution. In contrast, in the same example optimal put option execution is at T, by the complementary nature of put and call, and the smallest K for which $P(K) = 0$ equals $F(0,T)$. Hence, in this example the forward $F(0,T)$ is observable from $K \mapsto P(K)$ (and $K \mapsto C(K)$).

It is very difficult to construct a continuous-time pricing model that implies closed formulas for American-style call and put options. In the following paragraph 2.1, we construct one non-trivial, but quite academic, model of this form. Even though the model is quite simple, the American-style pricing formulas are already quite nasty compared to their European-style counterparts.

2.1 An American option example

We consider quite a simple stochastic stock price model, which allows to compute American-style option price formulas in closed form. To this end, we assume that $r(t) \equiv r \in \mathbb{R}$ and

$$S_t := S_0\, e^{(r+\lambda-\delta)\,t}\, 1_{\{\tau>t\}}, \quad t \geq 0,$$

where τ is assumed to be an exponential random variable with rate $\lambda \geq 0$. Due to the lack-of-memory property of the exponential law, $\{1_{\{\tau>t\}}\}_{t\geq 0}$ is a continuous-time Markov chain that changes its state only once at τ. This implies that the natural filtration of $\{S_t\}_{t\geq 0}$, which is assumed to be the market filtration, is given by

$$\mathcal{F}_t = \big\{\emptyset, \Omega, \{\tau > t\}, \{\tau \leq t\}\big\}.$$

Lemma 2.2 (American-style option prices)
Let $K > 0$. The American-style call option price is given by $C(K) = (S_0 - K)_+$, if $r + \lambda \leq 0$, and for $r + \lambda > 0$ by $C(K) =$

$$\begin{cases} (S_0 - K)_+ & , \delta > 0 \text{ and } \frac{S_0}{K} \geq U_C \\ \Big(S_0\, e^{-\delta\,T} - K\, e^{-(r+\lambda)\,T}\Big)_+, & \Big(\delta = 0\Big) \text{ or } \Big(\delta > 0 \text{ and } \frac{S_0}{K} \leq L_C\Big) \\ \left(S_0 \left(\frac{\delta\, S_0}{(r+\lambda)\,K}\right)^{\frac{\delta}{r+\lambda-\delta}} - K \left(\frac{\delta\, S_0}{(r+\lambda)\,K}\right)^{\frac{r+\lambda}{r+\lambda-\delta}}\right)_+ & , \text{else} \end{cases},$$

where

$$U_C := \max\Big\{\frac{1 - e^{-(r+\lambda)\,T}}{1 - e^{-\delta\,T}},\, \frac{r+\lambda}{\delta}\Big\},$$
$$L_C := U_C \cdot \min\Big\{1,\, e^{-(r+\lambda-\delta)\,T}\Big\}.$$

The American-style put option price is $P(K) = \max\{A, B, C\}$, with

$$A := \lambda\, K \int_0^T e^{-(r+\lambda)\,t}\,\mathrm{d}t + \Big(K\, e^{-(r+\lambda)\,T} - S_0\, e^{-\delta\,T}\Big)_+,$$

$$B := \begin{cases} K\, e^{-r\,T}\Big(1 - e^{-\lambda\,T}\Big) & , r \leq \frac{\lambda}{e^{\lambda\,T}-1} \\ \frac{\lambda\,K}{r+\lambda}\left(\frac{r}{\lambda+r}\right)^{\frac{r}{\lambda}} & , \text{else} \end{cases},$$

$$C := \begin{cases} K\, e^{-r\,T} - S_0\, e^{-\delta\,T} & , r \leq 0 \\ D & , \text{else} \end{cases}, \quad \text{where}$$

$$D := \begin{cases} K\, e^{-r\,T} - S_0\, e^{-\delta\,T} & , \delta > 0 \text{ and } \frac{S_0}{K} \geq U_P \\ K - S_0 & , \Big(\delta = 0\Big) \text{ or } \Big(\delta > 0 \text{ and } \frac{S_0}{K} \leq L_P\Big) \\ K\left(\frac{r\,K}{\delta\,S_0}\right)^{\frac{r}{\delta-r}} - S_0\left(\frac{r\,K}{\delta\,S_0}\right)^{\frac{\delta}{\delta-r}} & , \text{else} \end{cases},$$

and

$$L_P := \min\Big\{\frac{1 - e^{-r\,T}}{1 - e^{-\delta\,T}},\, \frac{r}{\delta}\Big\}, \quad U_P := L_P \cdot \max\Big\{1,\, e^{-(r-\delta)\,T}\Big\}.$$

Proof
See the Appendix. □

3 The equity forward

As mentioned earlier, the unique root K_* of the function $K \mapsto C(K) - P(K)$ equals the equity forward $F(0,T)$ in the case of European-style options. Also in the case of American-style options, K_* exists and might serve as a convenient source of information to approximate the required model quantity $\delta(.)$. The early exercise premium representations of American-style call and put options allow us to represent the equity forward $F(0,T)$ in terms of K_* as

$$F(0,T) = K_* - \mathbb{E}_{\mathbb{Q}}\Big[\int_0^T \big(\delta(s)\, S_s - K_*\, r(s)\big)\, e^{\int_s^T r(u)\,\mathrm{d}u} \times \\ \times \big(1_{\{S_s \geq E_c(s)\}} + 1_{\{S_s \leq E_p(s)\}}\big)\,\mathrm{d}s\Big]. \quad (4)$$

Consequently, the (unobserved) forward $F(0,T)$ and the observed root K_* differ by an expectation value that can be negative or positive or zero in general. In particular, the explicit example in paragraph 2.1 may be used to see this.

Example 3.1 ($F(0,T) > K_*$ and $F(0,T) < K_*$ possible)
In the example of paragraph 2.1, suppose $S_0 = 100$, $T = 3.5$, and denote by K_* the unique root of the function $f(K) = C(K) - P(K)$. If we set the parameters to $r = 0.1$, $\delta = 0$, and $\lambda = 0.2$, then we find $K_* = 127.663 < 141.9068 = F(0,T)$. Further, if the parameters are chosen as $r = -0.01$, $\delta = 0.09$, and $\lambda = 0.05$, then $K_* = 84.976 > 70.4688 = F(0,T)$. Summarizing, in general $K_* \neq F(0,T)$ and we may encounter the case $K_* > F(0,T)$ as well as $K_* < F(0,T)$.

Formula (4) at least shows that the observable root K_* typically lies somewhere "in the region" of $F(0,T)$, since the involved expectation value is usually small compared to K_* and $F(0,T)$. The observable root K_* can be bounded as follows.

Lemma 3.2 (Bounds on K_*)
Let $C(K)$ and $P(K)$ denote arbitrage-free prices of American-style call and put options, and let K_* satisfy $C(K_*) = P(K_*)$. Then K_* is bounded according to

$$F(0,T) \min_{t \in [0,T]} \Big\{e^{-\int_t^T r(s)\,\mathrm{d}s}\Big\} \leq K_* \leq S_0 \max_{t\in[0,T]} \Big\{e^{\int_0^t r(s)\,\mathrm{d}s}\Big\}.$$

Proof
Denote τ_P and τ_C the optimal stopping times for the put and the call with strike K_* and maturity T, respectively. Then,

$$\begin{aligned} 0 = C(K_*) - P(K_*) &= \mathbb{E}\Big[e^{-\int_0^{\tau_C} r(s)\,\mathrm{d}s}\,\big(S_{\tau_C} - K_*\big)_+\Big] \\ &\quad - \mathbb{E}\Big[e^{-\int_0^{\tau_P} r(s)\,\mathrm{d}s}\,\big(K_* - S_{\tau_P}\big)_+\Big] \\ &\begin{cases} \leq \mathbb{E}\Big[e^{-\int_0^{\tau_C} r(s)\,\mathrm{d}s}\,\big(S_{\tau_C} - K_*\big)\Big] & \text{(suboptimal put exercise)} \\ \geq \mathbb{E}\Big[e^{-\int_0^{\tau_P} r(s)\,\mathrm{d}s}\,\big(S_{\tau_P} - K_*\big)\Big] & \text{(suboptimal call exercise)} \end{cases}. \end{aligned}$$

These two estimates imply the bounds

$$\frac{\mathbb{E}\Big[e^{-\int_0^{\tau_P} r(s)\,\mathrm{d}s}\,S_{\tau_P}\Big]}{\mathbb{E}\Big[e^{-\int_0^{\tau_P} r(s)\,\mathrm{d}s}\Big]} \leq K_* \leq \frac{\mathbb{E}\Big[e^{-\int_0^{\tau_C} r(s)\,\mathrm{d}s}\,S_{\tau_C}\Big]}{\mathbb{E}\Big[e^{-\int_0^{\tau_C} r(s)\,\mathrm{d}s}\Big]}.$$

Since the discounted stock price has decreasing expectation by the supermartingale-property, we obtain the claimed bounds for K_* via estimating the supermartingale's expectation at τ_P (at τ_C) by the expectation at T (at 0). □

The bounds in Lemma 3.2 are sharp in the sense that one can construct an example in which K_* equals the lower bound and in which it equals the upper bound. Furthermore, Lemma 3.2 implies an observable upper bound for $F(0,T)$ in terms of K_* and $r(.)$. However, it does not give any information about a lower bound on $F(0,T)$. There is one trivial situation, in which we do observe information on a lower bound of $F(0,T)$.

Example 3.3 (Special (pathological) case)
If $P(\tilde{K}) = 0$ for some $\tilde{K} \in (0,\infty)$, and hence for all $K \leq \tilde{K}$, it follows that $\min\{S_t \,:\, t \in [0,T]\} \geq \tilde{K}$ almost surely, hence $F(0,T) = \mathbb{E}[S_T] \geq \tilde{K}$ as well. Unfortunately, however, the observed function $K \mapsto P(K)$ of put options usually does not have a root in $(0,\infty)$ in practice.

It appears to be an interesting open question whether the function $\delta(.)$ is model-free, like this is the case for European-style options. More concretely, do the current stock price S_0 and the functions $K \mapsto C(K)$ and $K \mapsto P(K)$ of American-style call and put prices for all strikes but one fixed maturity T uniquely determine the value δ (when assuming $\delta(.) \equiv$ constant)? An equivalent question is: can we construct two stock price models with same inital stock price S_0 but different values δ that imply precisely the same American-style call and put prices for all strikes and fixed maturity T?

4 Conclusion

Prominent literature dealing with the pricing of American-style options has been reviewed. Furthermore, it has been demonstrated that the problem of determining the equity forward from observations of American-style put and call option prices is highly non-trivial, unlike in the case of European-style options.

Appendix: Proof of Lemma 2.2

Obviously, the only non-deterministic stopping time with respect to the market filtration is τ, but $\min\{\tau, T\}$ is clearly a suboptimal strategy for the call option. Hence, $C(0) = S_0$ and for $K > 0$ we distinguish the following cases, introducing the notation $f(t) := S_0\, e^{-\delta\, t} - e^{-(r+\lambda)\, t}\, K$:

(1) $\delta = 0$:

(1.1) $r + \lambda = 0$:
Obviously, $C(K) = f(0)_+$.

(1.2) $r + \lambda \neq 0$:
We see $f^{'}(t) < 0$ if and only if $r + \lambda < 0$, so that

$$C(K) = \begin{cases} f(0)_+ & \text{, if } r+\lambda < 0 \\ f(T)_+ & \text{, if } r+\lambda > 0 \end{cases}$$

(2) $\delta > 0$:

(2.1) $r + \lambda \leq 0$:
The function $f(t)$ is decreasing, so $C(K) = f(0)_+$.

(2.2) $0 < r + \lambda < \delta$:
The function $f(t)$ is decreasing for $t \leq t_*$ and increasing thereafter, where

$$t_* := \frac{\log\left(\frac{K\,(r+\lambda)}{\delta\,S_0}\right)}{r + \lambda - \delta}.$$

Furthermore,

$$t_* > 0 \;\Leftrightarrow\; r + \lambda < \delta\,\frac{S_0}{K},$$

which implies

$$C(K) = \begin{cases} \max\{f(0), f(T)\}_+ & , r + \lambda < \delta\,\frac{S_0}{K} \\ f(T)_+ & , \text{else} \end{cases}.$$

Since $g_T(x) := (1 - \exp(-x\,T))/x$ is a decreasing function in $x \geq 0$,

$$\frac{r + \lambda}{\delta} \leq \frac{1 - e^{-(r+\lambda)\,T}}{1 - e^{-\delta\,T}}.$$

Furthermore, $f(T) \geq f(0)$ if and only if

$$\frac{S_0}{K} \leq \frac{1 - e^{-(r+\lambda)\,T}}{1 - e^{-\delta\,T}}.$$

Summarizing,

$$C(K) = \begin{cases} f(T)_+ & , \frac{S_0}{K} \leq \frac{1 - e^{-(r+\lambda)\,T}}{1 - e^{-\delta\,T}} \\ f(0)_+ & , \text{else} \end{cases}.$$

(2.3) $r + \lambda = \delta$:
Obviously, $C(K) = f(0)_+$.

(2.4) $r + \lambda > \delta$:
The function $f(t)$ is increasing for $t < t_*$ and decreasing thereafter, where t_* is the same as in case (2.2). Furthermore, we have

$$t_* > 0 \Leftrightarrow r + \lambda > \delta\,\frac{S_0}{K},$$
$$t_* < T \Leftrightarrow r + \lambda < \delta\,\frac{S_0}{K}\,e^{(r+\lambda-\delta)\,T},$$

which implies

$$C(K) = \begin{cases} f(0)_+ & , r + \lambda \leq \delta\,\frac{S_0}{K} \\ f(T)_+ & , r + \lambda \geq \delta\,\frac{S_0}{K}\,e^{(r+\lambda-\delta)\,T} \\ f(t_*)_+ & , \text{else} \end{cases}.$$

Putting together all cases, we end up at the claimed formula. When summarizing the cases, it is helpful to make use of the relation

$$r + \lambda \geq \delta \;\Leftrightarrow\; \frac{r + \lambda}{\delta} \geq \frac{1 - e^{-(r+\lambda)\,T}}{1 - e^{-\delta\,T}},$$

which is true for $r+\lambda>0$ (and $\delta\geq 0$), since $g_T(x)$ from case (2.2) is decreasing in $x\geq 0$.
For the put option, the stochastic stopping rule $\min\{\tau,T\}$ may or may not be optimal. We denote the expected, discounted payoff under the strategy $\min\{\tau,T\}$ by A and see

$$\begin{aligned}A &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r\,\min\{\tau,T\}}\,(K-S_{\min\{\tau,T\}})_+\right] = \mathbb{E}_{\mathbb{Q}}\left[1_{\{\tau\leq T\}}\,e^{-r\,\tau}\,K\right]\\ &\quad + \mathbb{E}_{\mathbb{Q}}\left[1_{\{\tau>T\}}\,e^{-r\,T}\left(K-S_0\,e^{(r+\lambda-\delta)\,T}\right)_+\right]\\ &= \lambda\,K\int_0^T e^{-(r+\lambda)\,t}\,\mathrm{d}t + \left(K\,e^{-(r+\lambda)\,T}-S_0\,e^{-\delta\,T}\right)_+.\end{aligned}$$

In addition to this possible strategy every determinstic stopping time $t\in[0,T]$ is an element of $\mathcal{T}[0,T]$. Hence, $P(K)$ equals the maximum of A and

$$\begin{aligned}&\sup_{t\in[0,T]}\left\{\mathbb{E}_{\mathbb{Q}}\left[e^{-r\,t}\,(K-S_t)_+\right]\right\}\\ &= \sup_{t\in[0,T]}\left\{K\,e^{-r\,t}\left(1-e^{-\lambda\,t}\right)+\left(K\,e^{-(r+\lambda)\,t}-S_0\,e^{-\delta\,t}\right)_+\right\}\\ &= \sup_{t\in[0,T]}\left\{\max\left\{K\,e^{-r\,t}\left(1-e^{-\lambda\,t}\right),\,K\,e^{-r\,t}-S_0\,e^{-\delta\,t}\right\}\right\}\\ &= \max\left\{\sup_{t\in[0,T]}\left\{K\,e^{-r\,t}\left(1-e^{-\lambda\,t}\right)\right\},\,\sup_{t\in[0,T]}\left\{K\,e^{-r\,t}-S_0\,e^{-\delta\,t}\right\}\right\}.\end{aligned}$$

We denote the two inner suprema by B and C, respectively, and observe

$$B = \begin{cases} K\,e^{-r\,T}\left(1-e^{-\lambda\,T}\right) & \text{, } r\leq \frac{\lambda}{e^{\lambda\,T}-1} \\ \frac{\lambda\,K}{r+\lambda}\left(\frac{r}{\lambda+r}\right)^{\frac{r}{\lambda}} & \text{, else}\end{cases}.$$

For $r\leq 0$, C is obviously maximized at T, while for $r>0$ we have $C=D$, which we can compute by distinguishing several cases, denoting $f(t):=K\,e^{-r\,t}-S_0\,e^{-\delta\,t}$:

(1) $\delta=0$:
Obviously, $f(t)$ is decreasing, so $D=f(0)$.

(2) $\delta>0$:

(2.1) $0<r<\delta$:
The function $f(t)$ is increasing for $t<t_*$ and decreasing thereafter, where

$$t_* := \frac{\log\left(\frac{\delta\,S_0}{r\,K}\right)}{\delta-r}.$$

Furthermore, the maximum t_* is within $(0,T)$ if and only if

$$S_0\,\frac{\delta}{r}\,e^{(r-\delta)\,T} < K < S_0\,\frac{\delta}{r}.$$

This implies

$$D = \begin{cases} f(0) & \text{, } K\geq S_0\,\frac{\delta}{r} \\ f(T) & \text{, } K\leq S_0\,\frac{\delta}{r}\,e^{(r-\delta)\,T} \\ f(t_*) & \text{, else}\end{cases}.$$

(2.2) $r = \delta$:
The function $f(t)$ increases (decreases) if $K < S_0$ ($K > S_0$), so that

$$D = \begin{cases} f(0) & , K \geq S_0 \\ f(T) & , \text{else} \end{cases}.$$

(2.3) $r > \delta$:
The function $f(t)$ decreases for $t < t_*$ and increases thereafter, where t_* is the same as in case (2.1). Furthermore,

$$t_* > 0 \Leftrightarrow K > S_0 \frac{\delta}{r},$$

which implies

$$D = \begin{cases} f(T) & , K \leq S_0 \frac{\delta}{r} \\ \max\{f(0), f(T)\} & , \text{else} \end{cases}.$$

It remains to check what happens in the case $K > S_0\,\delta/r$. Like in case (2.2) of the call option, it follows that

$$\frac{r}{\delta} \geq \frac{1 - e^{-r\,T}}{1 - e^{-\delta\,T}}.$$

Furthermore, we observe that

$$f(T) \geq f(0) \Leftrightarrow \frac{S_0}{K} \geq \frac{1 - e^{-r\,T}}{1 - e^{-\delta\,T}}.$$

Summarizing,

$$D = \begin{cases} f(T) & , \frac{S_0}{K} \geq \frac{1-e^{-r\,T}}{1-e^{-\delta\,T}} \\ f(0) & , \text{else} \end{cases}.$$

When summarizing the cases for D, it is helpful to make use of the relation

$$r \geq \delta \Leftrightarrow \frac{r}{\delta} \geq \frac{1 - e^{-r\,T}}{1 - e^{-\delta\,T}},$$

which is true for $r > 0$ (and $\delta \geq 0$), and follows from decreasingness in $x \geq 0$ of the function $g_T(x)$ from case (2.2) of the call option.

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