XAIA INVESTMENT

Functional Relationships between Stock Prices and CDS Spreads

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Abstract We aim to verify statistically if the often proposed hyperbolic relationship between stock price and default intensity is evident in real-world data. CDS spreads are used as proxy for the unobservable default intensity. Although the relationship can be seen clearly in plots, standard regression models are unable to give reliable parameter estimates. We conclude that the hyperbolic relation is a fair enough proxy for the real functional relation in the context of credit-equity modeling, but if the aim is to give an estimate for the CDS spread based on observations of the stock price, other statistical methods should be used.

1 Introduction

In order to price both bonds / credit derivatives and equity derivatives, one needs to model not only the stock price but also the probability that the reference entity defaults. This can be achieved with structural models (which rely on fundamentals) or with reduced form models (where only the necessary variables are modeled stochastically).

The reduced form $1\frac{1}{2}$-factor model offers a good compromise between theory and practice: The stock price S_t is modeled as a stochastic process, and the default intensity λ_t (from which we can deduce the firm's default probability) is expressed as a function of the stock price. On the one hand this is sophisticated enough to display many of the desired theoretical and practical properties of a credit-equity model as stated in Mai (2012), on the other hand it is easy enough to be implemented in practice. For a list of references regarding $1\frac{1}{2}$-factor models the interested reader is referred to Mai (2012).

In this context, a hyperbolic relationship between stock price and default intensity is often proposed:

$$\lambda_t = \lambda_0 \left(\frac{S_t}{S_0}\right)^a = \tilde{\lambda}_0 S_t^a, \tag{1}$$

with model parameters $\lambda_0 > 0$, $\tilde{\lambda}_0 = \lambda_0 S_0^{-a}$ and $a < 0$.

We aim to verify statistically if a functional relationship of the form (1) is present in real-world data, exemplarily using data of seven names in our portfolio XAIA Credit Debt Capital. Since the default intensity is a fictitious quantity and cannot be observed directly in the market, CDS spreads are used as proxy. This proxy may be justified by the simplest model for the time of default τ, where we assume $\tau \sim \exp(\lambda)$ is exponentially distributed with some constant parameter λ that represents the default intensity. In this context, a proxy for the CDS spread is given by $\lambda(1-R)$, where R denotes the recovery rate. This implies that the default intensity may be viewed as approximately linear in the CDS spread and vice versa. For more information on this proxy see, e.g., Hull (2009), p. 500. We choose 1-year CDS spreads, as this is the shortest CDS maturity for which quotes are available.

2 Simple graphical tests

As a first step, one could have a look at plots of the CDS spread vs. the stock price. In our sample, three patterns emerged in the stock vs. CDS plots:

(i) For several entities the hyperbolic relationship is clearly visible in the plot, e.g. series 1 in Fig. 1.

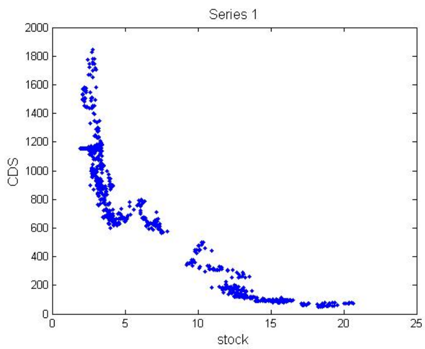

Fig. 1: Clearly visible hyperbolic relationship between stock price and CDS.

(ii) For others, the hyperbolic relation is present, but has some sort of "tail", e.g. series 2 in Fig. 2.

(iii) For the rest, there is no visible relationship between stock price and CDS - at first. See series 3 in Fig. 2.

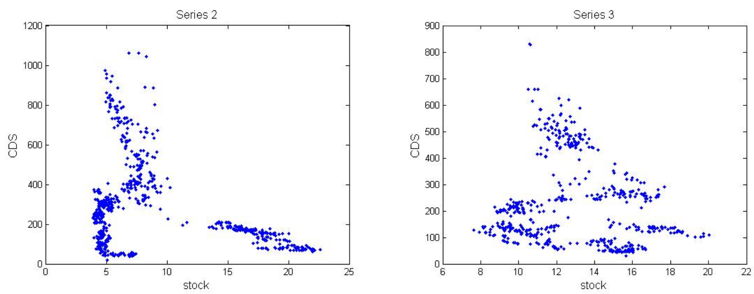

Fig. 2: Hyperbolas with "tail" (left) and chaotic relations (right).

For those entities in our sample, where we don't find the desired hyperbolic relation firsthand, it is possible to find shorter time intervals in the observation period where it does hold. We use color coded versions of the plots in Fig. 2 to observe how the relation between stock price and CDS changes over time, cf. Fig. 3. For the hyperbola with "tail" (series 2, Fig. 2 and 3 left), the color code shows that the aberrant points in the "tail" (green) were observed in a later time interval, so the hyperbolic relation holds up to some time point. For series 3 (Fig. 2 and 3 right), where no relation is detectable firsthand, the color code enables us to find some order in the chaos: At first, stock and CDS follow a hyperbola, then there is a transition period (green) where the

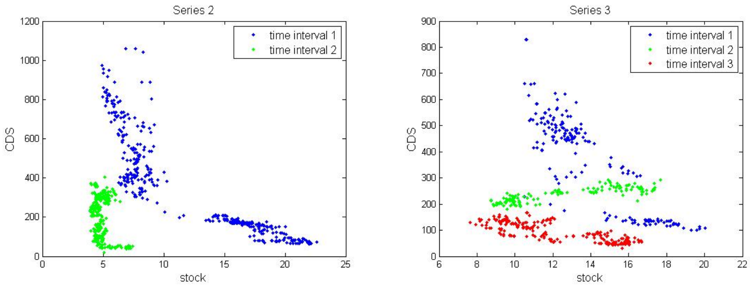

Fig. 3: Splitting the series into shorter intervals shows how the functional relation changes over time.

functional relationship is completely different, and afterwards we find again a hyperbolic relation, but a different one. Similar observations are made in all seven studied cases, also the ones that are not depicted in the graphs.

3 Mathematical verification: An attempt via regression

The observed functional relationship should be verified with the help of a mathematical model. Applying the natural logarithm to both sides of equation (1) yields a linear relationship:

$$\ln(\lambda_t) = \ln(\lambda_0) + a(\ln(S_t) - \ln(S_0)) = \ln(\tilde{\lambda}_0) + a\ln(S_t). \quad (2)$$

A first check if $\ln(\lambda_t)$ and $\ln(S_t)$ could be linearly related is to compute the sample correlation and see if it is sufficiently high.
The natural choice of model would be a linear regression model for the logarithmic series:

$$Y_t = \beta_0 + \beta_1 X_t + \epsilon_t \quad \forall t, \quad (3)$$

with coefficients $\beta = (\beta_0, \beta_1)$, $Y_t = \ln(\lambda_t)$, $X_t = \ln(S_t)$ and ϵ_t a series of error terms.

A short review of time-series least squares regression

Typically the model is fitted via ordinary least squares (OLS), i.e. the sum of the squared distances between the observed points (X_t, Y_t) and the points on the regression line $(X_t, \hat{Y}_t)$ is minimized. Here, $\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t$ denote the values fitted via (3) and $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1)$ are the OLS estimates for β:

$$\begin{aligned} \hat{\beta}_0 &= \frac{\sum_{t=1}^n X_t^2 - n\bar{X}\sum_{t=1}^n X_t Y_t}{\sum_{t=1} X_t^2 - n\bar{X}^2}, \\ \hat{\beta}_1 &= \frac{\sum_{t=1}^n X_t Y_t - n\bar{X}\bar{Y}}{\sum_{t=1}^n X_t^2 - n\bar{X}^2}, \end{aligned} \quad (4)$$

where $\bar{X} = \frac{1}{n}\sum_{t=1}^n X_t$, $\bar{Y} = \frac{1}{n}\sum_{t=1}^n Y_t$ are the sample means of the two series.
In the time series regression framework, the following assumptions are usually made, see Wooldridge (2009), p. 370f:

(1) True model is linear: it can be written as in (3)

(2) No multicollinearity in the case of multiple regressors: none of the regressors is a linear combination of the others

(3) X is strictly exogenous: $\mathbb{E}[\epsilon_t|X] = 0$, where X is the vector of the X_t for all times t

(4) Homoskedasticity: $\mathbb{V}[\epsilon_t|X] = \mathbb{V}[\epsilon_t] = \sigma_\epsilon^2$, where $\mathbb{V}$ denotes the variance operator

(5) No autocorrelation of the errors: Corr$[\epsilon_t, \epsilon_s|X] = 0$ for $t \neq s$

(6) Independently and identically normally distributed error terms: $\epsilon_t \sim N(0, \sigma_\epsilon^2) \quad \forall t$

Assumptions 1-3 ensure that the vector of OLS estimates $\hat{\beta}$ is unbiased. Under assumptions 1-5, $\hat{\beta}$ is the best linear unbiased estimate for β. This means that $\mathbb{E}[\hat{\beta}] = \beta$ (unbiased), $\hat{\beta}$ is a linear function of the observations of Y, and it has the smallest variance among all such estimates. Including the normality of errors assumption (which implies assumptions 3-5), $\hat{\beta}$ even is the best unbiased estimate, i.e the unbiased estimate with the smallest variance. Further, the usual coefficient significance tests and confidence intervals are valid.

Checking the assumptions in our framework

Some of the assumptions (1) - (6) are very restrictive. Indeed, if the proposed regression is conducted with our data, we find strong autocorrelations of the error terms in all of the cases. The Breusch-Pagan test, see Wooldridge (2009), p. 273, shows that the assumption of homoscedasticity may not be valid as well. Both imply that the assumption (6) of independent identically distributed normal error terms cannot hold. The strict exogeneity of X is also questionable: There exists extensive literature on the relation between stock and CDS markets, partially summarized in Schempp (2013), and several authors, including Forte, Lovreta (2012) , find that stock markets do not always lead CDS markets, but sometimes also the converse is observable. Most likely there exists feedback between the two markets, which invalidates assumption (3). The consequences of these findings are that the OLS estimates obtained from our regression are biased and no longer of minimal variance. Significance tests for the coefficients and confidence intervals are no longer valid. But since we did not want to obtain parameter estimates, this need not bother us.

Goodness of fit?

Even if we do not want to obtain reliable parameter estimates, some goodness-of-fit measure would be nice. The classical one in OLS regression is the coefficient of determination R^2, which is defined as follows:

$$R^2 := 1 - \frac{\sum_{t=1}^n (Y_t - \hat{Y}_t)^2}{\sum_{t=1}^n (Y_t - \bar{Y})^2}, \tag{5}$$

where $\bar{Y}$ is the sample mean of the Y_t.
This measure is still valid in case of heteroscedasticity and autocorrelation - but only if the underlying time series are stationary[1] and weakly dependent[2].
Unfortunately it can be seen immediately that none of the time

[1] A time series $(X_t)_{t\in\mathcal{N}}$ is stationary if for any $n \in \mathcal{N}$, $t_1 \dots t_n$ and $h \geq 1$ the distributions of $(X_{t_1} \dots X_{t_n})$ and $(X_{t_1+h} \dots X_{t_n+h})$ are equal. In case we have $\mathbb{E}[X_t^2] < \infty$, this implies the weaker notion of second order stationarity, where we have $\mathbb{E}[X_t]$ and $\mathbb{V}[X_t]$ constant and the covariance function $\mathrm{Cov}[X_t, X_{t+h}]$ depends only on h.

[2] A time series is weakly dependent if Corr$[X_t, X_{t+h}] \overset{h\to\infty}{\to} 0$.

series in our sample satisfies even the properties for second order stationarity. They all exhibit some form of time trend, so the expectation is not constant and the R^2 is not a valid goodness-of-fit measure. According to Wooldridge (2009), in the presence of time trends the R^2 might systematically overestimate the dependence between Y_t and X_t, as it is not clear that the trend in X_t really drives the trend in Y_t. They might be correlated, but this does not imply causality: Both trends might be induced by other unobserved quantities.
Since we cannot rely on the R^2, we will assess the goodness-of-fit of regression (3) only graphically.

An alternative model that accounts for the trends

We noted above that none of our data series was stationary, each exhibited some form of time trend. Consistent with the Black-Scholes model for stock prices,

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad (W_t)_{t\geq 0} \text{ Brownian Motion,} \tag{6}$$

where the unconditional expectation of S_t is given by $S_0 \exp(\mu t)$, we try to include a linear time trend in our logarithmic model:

$$Y_t = \beta_0 + \beta_1 X_t + \beta_2 t + \epsilon_t, \tag{7}$$

where $t \in \{1 \ldots T\}$ and T is the number of discrete observations. According to Wooldridge (2009), the estimate for β_1 is the same here as in the case where we first detrend Y_t and X_t (using a linear trend!) and then run a regression on the detrended series. Often a quadratic time trend looks more appropriate for both Y_t and X_t, so we try this as well. Running these regressions on our data, we still find that the residuals are autocorrelated and heteroscedastic, so again parameter estimates and t-statistics are not reliable. Wooldridge (2009) proposes an altered R^2 as goodness-of-fit measure that accounts for time trends, but since this is not useful for comparing the models with and without time trend, we will stick to assessing the fit graphically.

Graphical goodness-of-fit of the regression models

First let's have a look at the model without time trend: In the case (i), where the hyperbolic relationship is already evident in the stock-CDS-plots, we get a nice fit, cf. Fig. 4.

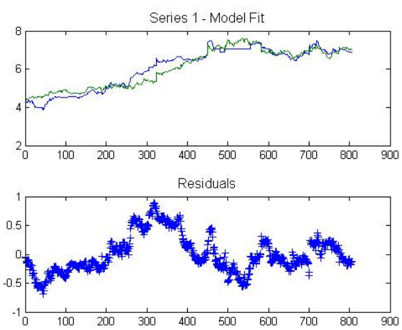

Fig. 4: Nice fit in case (i), where the hyperbolic relation is clearly visible: Model fit (green) compared to real-world data (blue) and residuals.

For the other cases (ii) and (iii), double-logarithmic regression does not achieve a good fit when using all available points. Considering only the shorter time period where the relation presumably holds, one can achieve a significantly better fit, cf. Fig. 5.

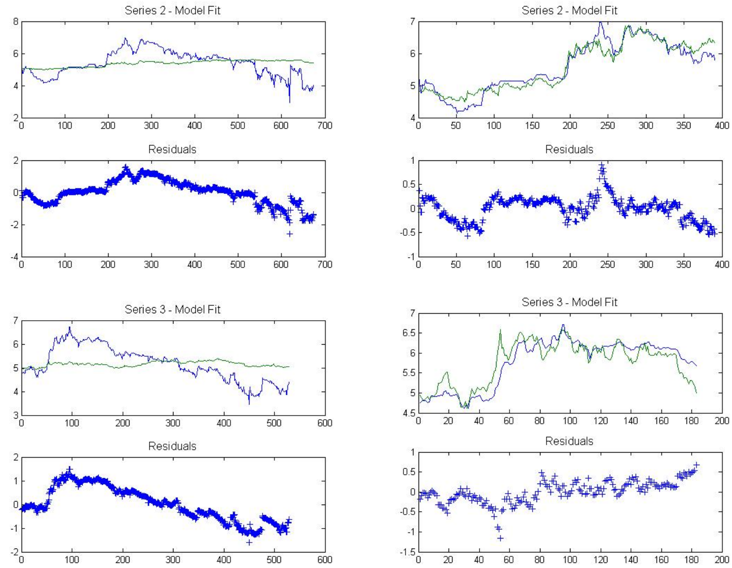

Fig. 5: Significant improvement of fit when restricting to periods where the hyperbola is observable: Model fit (green) compared to real-world data (blue).

For the model with time trend (7), we find that including it does not change much in the case (i), where the first model already worked well, cf. Fig. 6.

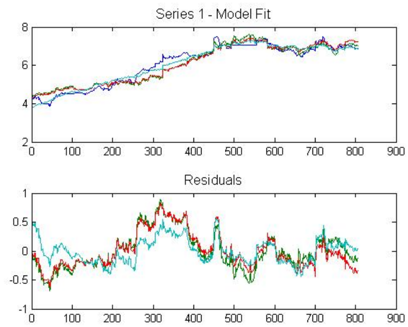

Fig. 6: Models with linear (red) or quadratic (turquois) time trend do not offer much improvement.

But looking at the cases (ii) and (iii), where the hyperbolic relation was not evident at first sight, including the time trend significantly improves the fit over the whole observation period, cf. Fig. 7.

4 Conclusion and outlook

Graphical tests and double logarithmic regression encourage our assumption that the relationship between stock prices and CDS spreads can often be modeled using a hyberbolic function. Yet this cannot be confirmed formally as the assumptions in the classical regression framework do not hold. Relying on graphical

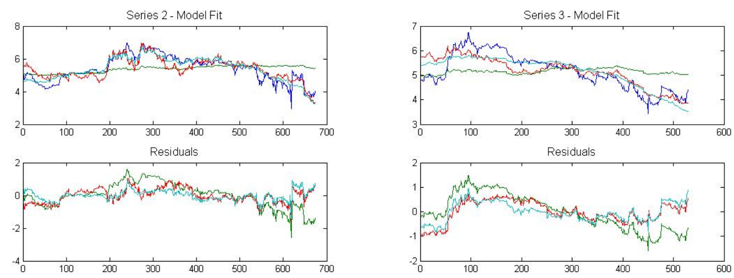

Fig. 7: Including a linear (red) or quadratic (turquois) time trend, a significantly better fit is possible in cases (ii) and (iii).

tests, we conclude that the functional relationship specified above offers a fair enough proxy for the real functional dependency in many cases. Sometimes it is necessary to restrain oneself to a shorter observation period, alternatively inclusion of a time trend could improve the fit. As previously stated, we usually won't receive good parameter estimates from the regression, and goodness-of-fit tests do not give reliable outputs, therefore this regression should not be used for parameter estimation and quantification of the functional dependence.
Possible reasons why the regression assumptions and thus the parameter estimates fail in our framework are that most likely we did not include all explanatory variables in the regression, or that we did not use the real dependent variable, the unobservable default intensity, but the CDS spread as a proxy for it. If the aim is to give a measure of functional dependence between stock price and CDS spread, the next step in the regression framework would be a finite distributed lag model, where lagged versions of X_t enter the model as additional regressors. According to Forte, Lovreta (2012), the market standard is to relate stock returns and changes in CDS spreads via a vector autoregressive model, but this does not suit the simplicity requirements in reduced form models that are typically used for pricing and hedging purposes.

References

Forte, S., Lovreta, L.: Time-varying credit risk discovery in the stock and CDS markets: Evidence from quiet and crisis times, *European Financial Management, Forthcoming*

Hull, J.C.: Options, futures and other derivatives, 7th edition, 2009

Mai, J.-F.: The joint modeling of debt and equity: An introduction, *XAIA Homepage article, 2012*

Schempp, P.: A review of the empirical interrelations among the CDS, bond and stock market, *XAIA Homepage article, 2013*

Wooldridge, J.M.: Introductory econometrics, 4th edition, 2009