Abstract

The aim of this document is to describe the basics of discount curve bootstrapping and to provide some insights into the underlying principles. It is meant to be a comprehensive explanation of the standard techniques used when extracting discount factors from market data. In particular, we present the newly established standards necessary to consistently price swaps with longer tenors or payments/swaps in different currencies.

1 Introduction

The aim of a bootstrapping procedure is to extract discount factors from market quotes of traded products. Stated differently, we want to derive “the value of a fixed payment in the future”, respectively “the fixed future repayment of money borrowed today”, from market prices. Therefore, we always have to restrict our focus to a specific investment universe consisting of the elements we want to consider. Here, we focus on bootstrapping from swap curves, whereas the procedure for bond curves is slightly different. It is shown what the underlying assumptions are and how these procedures are based on static no-arbitrage considerations.

In the following, we will focus on the extraction of discount factors from European overnight index swaps (OIS), called EONIA swaps. There are several reasons for that:

- During the financial crisis, the spread between OIS and EURIBOR/LIBOR rates with longer tenor significantly increased as the previously neglected risks related to unsecured bank lending became apparent. Consequently, OIS rates can now be considered the best available proxy for risk-free rates (see e.g. Hull and White 2012). For academics such as Hull and White this is a strong argument why those rates have to be used when valuing derivatives.

- Furthermore, more and more derivative contracts are collateralized and this trend will be reinforced by current regulatory changes (EMIR in Europe, Dodd-Frank in the US). Especially in the inter-dealer market (origin of the quoted prices published by the inter-dealer brokers) most trades are collateralized. There are several papers (see e.g. Piterbarg (2010) or Fuji et al. (2010a)) arguing that for the valuation of those trades, discount factors from OIS swaps have to be used, since overnight rates can be earned/have to be paid on cash collateral. For that reason, it is standard now in the inter-dealer market to value swaps based on OIS discount factors.
Some clearing houses have changed their valuation procedures accordingly as well.

- The aim of the document is to give an introduction into the general bootstrapping procedure. This procedure is independent from the chosen reference rate respectively the corresponding swap universe. Consequently, the chosen special case can serve as an introduction to the concept in general.

Our aim is to compute \( D(t, T) \), the discount factor at time \( t \) for the horizon \( T \). In classical school mathematics with constant interest rate \( r \) and continuous compounding this quantity would be given as \( D(t, T) = \exp(-r(T-t)) \). There are many other ways to describe a yield curve, e.g. zero rates or forward rates, but they always depend on the underlying compounding and day-count conventions. Since most applications rely on discount factors anyway, we directly consider those.

2 One-curve bootstrap

2.1 Simple setup

Starting point of a bootstrap is the decision for a reference rate. As mentioned before, we will consider European overnight rates. Consequently, we are given:

- a "riskless" asset \( B \) with value process \( \{B(t)\}_{t \geq 0}, B(0) = 1 \). This asset mirrors an investment in the reference rate \( r_{OIS} \), i.e. continuously investing money into an asset paying the reference rate. Consequently,

\[
B(t) = \prod_{j=1}^{n_t} \left(1 + \frac{r_{OIS}^j d_j}{360}\right), \quad t \geq 0,
\]

where \( n_t \) is the number of business days in the considered period from today \( 0 \) until \( t \) and \( r_j \), the reference rate fixed on day \( j \) relevant for the next \( d_j \) days (usually one day as it is an overnight rate, except for weekends and holidays). The existence of such an asset is always implicitly assumed but sometimes not stated explicitly. However, it is THE central assumption of a bootstrap. We want to compute discount factors based on the assumption that on every date, one is able to borrow/invest at the given reference rate. It becomes clear here why using EURIBOR rates with longer tenor is difficult as it does not seem reasonable to receive those rates on a riskless, respectively default free, asset.

- a set of EONIA swaps\(^1\) with maturities \( T_1 < T_2 < \ldots < T_n \), e.g. \( T_i \in \{3m, 6m, 9m, 1y, 2y, \ldots, 30y\} \), respectively the corresponding par swap rates \( s_i \). We assume that a swap with maturity \( T_i \) exchanges payments at the dates \( T_1 < T_2 < \ldots < T_i \), which at \( T_j \) is a fixed payment of \( s_i(T_j - T_{j-1}) \) versus a floating payment\(^2\) of \( f(T_{j-1}, T_j)(T_j - T_{j-1}) \). The

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\(^1\)For more information regarding EONIA swaps, see e.g. http://www.euribor-ebf.eu/eoniaswap-org/about-eoniaswap.html.

\(^2\)For simplicity, we omit daycount conventions, assume both legs to have equal payment dates, different swaps to have coinciding payment dates on
floating payment is linked to the reference rate by

$$ f(T_{j-1}, T_j)(T_j - T_{j-1}) = \prod_{j=n_{T_j-1}+1}^{n_{T_j}} \left( 1 + \frac{r^{OIS}_j d_j}{360} \right) - 1. $$

The value of the sum of all fixed payments will be called $FixedLeg$, the value of the sum of all floating payments will be called $FloatingLeg$. Based on the definition of discount factors, it obviously holds that

$$ FixedLeg_i = s_i \sum_{j=1}^{i} D(0, T_j)(T_j - T_{j-1}). $$

There is an obvious connection between both aforementioned products, which is that

$$ f(T_{j-1}, T_j)(T_j - T_{j-1}) = \frac{B(T_j)}{B(T_{j-1})} - 1 $$

holds. That is, the swaps are structured in a way that the floating payments equal the interest earned on the reference rate. From that, we can deduce that an investment of 1 in $B$ at $t = 0$ can yield all the required floating payments plus an additional fixed payment of 1 at the last payment date. Consequently, the value of a floating leg with maturity $T_i$ is given by

$$ FloatingLeg_i = 1 - D(0, T_i). $$

Requiring a swap to have legs with the same value, as the swap can be entered with zero upfront costs, we end up with the following set of equations the discount factors with maturities $T_i$ should satisfy to match market prices:

$$ s_i \sum_{j=1}^{i} (T_j - T_{j-1}) D(0, T_j) = 1 - D(0, T_i), \quad 1 \leq i \leq n. \quad (1) $$

Iteratively solving for $D(0, T_i)$ by

$$ D(0, T_i) = \frac{1 - s_i \sum_{j=1}^{i-1} (T_j - T_{j-1}) D(0, T_j)}{1 + s_i (T_i - T_{i-1})}, \quad 1 \leq i \leq n, \quad (2) $$

is commonly known as bootstrapping the discount factors. This is also the formula we use, however, we would like to motivate it in a different way using direct replication arguments. The aim of this is to allow for a better understanding of the resulting prices. That is, we can motivate it by simple static replication arguments.

overlapping time intervals and furthermore, that longer-dated swaps extend the set of fixed leg payment dates of the previous swaps by one additional payment date. The first assumptions can be easily relaxed, to relax the last assumption we need additional methods which we will present in Section 2.3.
2.2 Intuitive interpretation

A discount factor represents today’s \((T_0 = 0)\) value of a fixed payment in the future. If we can replicate this payment structure by a portfolio consisting of the given instruments, the discount factor is given as the respective portfolio’s market value today. For the maturities \(T_i\) of the given swaps, such an approach can be applied. We will start from the smallest maturity and prove the rest by induction.

For \(T_1\), consider a portfolio consisting of an investment of 1 into the bank account \(B\) and a receiver swap with maturity \(T_1\). Buying a receiver swap corresponds to paying the floating leg (here: \(B(T_1)/B(T_0) - 1\)) in a swap contract and receiving the fixed leg (here: \(s_1(T_1 - T_0)\)). The payments of the respective portfolio can be found in Table 1, using \(^3 B(T_0) = B(0) = 1\). Consequently,

\[
D(0, T_1) = \frac{1}{1 + s_1(T_1 - T_0)}.
\]

This of course is in accordance with Equation (2).

Now, we can iteratively extend this procedure for the remaining maturities. Assume that we are able to replicate fixed payments at \(T_i\) for \(i < n\) and thus know \(D(0, T_i), i < n\). We are trying to replicate a fixed payment at \(T_n\). As before, invest 1 into the bank account and additionally buy a receiver swap with maturity \(T_n\). As mentioned earlier, the investment into the bank account can replicate the floating payments of the swap. Additionally, we will replicate the payments of the fixed leg up to \(T_{n-1}\) using the replicating portfolios we have by our “induction hypothesis”. The payments of the respective portfolio can be found in Table 2 (on the last page). In total, we found a portfolio that replicates a fixed payment of \(1 + s_n(T_n - T_{n-1})\) at \(T_n\), where today one has to pay \(1 - s_n \sum_{j=1}^{n-1} (T_j - T_{j-1}) D(0, T_j)\). Solving this for \(D(0, T_n)\) yields

\[
D(0, T_n) = \frac{1 - s_n \sum_{j=1}^{n-1} (T_j - T_{j-1}) D(0, T_j)}{1 + s_n (T_n - T_{n-1})},
\]

again being in accordance with Equation (2). To summarize, we have shown by an iterative argument that Equation (2) can be motivated by simple replication arguments. Unfortunately, this

\(^3\)Note that in practice, when constructing this replicating portfolio, there will take place an exchange of collateral during the lifetime of the transaction to account for a change of market value of the swap contract. However, assuming that we will have to pay EONIA on received collateral and receive EONIA on paid collateral, this has no impact on the final payments as we assumed we can invest/finance this collateral at EONIA (using \(B\)) as well.
approach as well as Equation (2) in general only hold for the discount factors with maturities $T_i$ and the case that additional swaps only add one extra payment date. However, it hopefully allows for a better intuitive understanding of bootstrapping procedures in general.

2.3 Implementation

In practice, the available data will in most cases not fulfill the ideal assumptions described in the previous paragraphs. For example, one might use other contracts than swaps on the short end or one might have “overlapping” contracts. However, we do not want to elaborate on the process of selecting the contracts deemed the most adequate or liquid. Another aspect consists of the fact that in the iterative procedure of solving Equation (2), there might be more than one unknown variable. This happens when payment dates of longer contracts do not coincide with those of shorter contracts or when a longer contract has several payment dates after the maturity of the previous contract. This is an important aspect we want to illustrate.

Assume that we have already bootstrapped the values $D(0, T_i), 1 \leq i \leq n,$ and now, we consider a swap with maturity $T$, swap rate $s$ and payment dates $t_1 < t_2 < \ldots < t_k = T$. To be consistent with market quotes, we are looking for discount factors $D(0, t_j), 1 \leq j \leq k,$ such that

$$s \sum_{j=1}^{k} (t_j - t_{j-1}) D(0, t_j) = 1 - D(0, T). \quad (3)$$

If $t_j \in \{T_1, \ldots, T_n\}$, the corresponding discount factor $D(0, t_j)$ is already given. Here, we consider the fact that more than one of those discount factors is unknown. Consequently, there is no unique solution to the problem at hand. Consider a $t_j$ such that there exists $T_i < t_j \leq T_{i+1}$. This is a "known" problem, as after a completed bootstrap we will also need discount factors for time points in between the others. One usually employs some kind of interpolation, not based on the discount factors directly but instead based on forward or zero rates describing those discount factors. As the discount factors used in the bootstrap should be the same as the discount factors used later for pricing purposes, it is clear that already in the bootstrapping procedure the desired interpolation scheme has to be used. Hagan, West (2006) argue similarly and compare existing interpolation approaches. The interested reader is referred to their paper for a full description of the related problems/considerations.

Consider now the remaining $t_j$ such that $t_j > T_n$. If there is only one of them, Equation (3) obviously has a unique solution as given in Equation (2). If there are several of them, the previous considerations regarding interpolation have to be applied again. That is, we only have to consider $D(0, T) = D(0, t_k)$ as an unknown variable whereas $D(0, t_j)$ with $T_n < t_j < T$ have to be computed via interpolation, all depending on the unknown variable $D(0, T)$. In that case, a solution to Equation (3), which is now an equation with one unknown variable $D(0, T)$, can not be found in closed form but instead must be found via numerical
3 Additional curves in the same currency (dual curve stripping)

Until the beginning of the recent financial crisis, bootstrapping discount factors from EURIBOR/LIBOR rates was sufficient as there was only a negligible spread between swaps with different tenor. However, this has changed considerably. The risk related to unsecured bank lending became apparent and consequently, the longer the tenor of EURIBOR/LIBOR rates, the higher the risk investors associate with those rates and thus, the higher the rates. It is not possible to value a swap with a different tenor based on the previously bootstrapped discount factors. For that, another procedure, the so called dual curve stripping or curve cooking is needed, which we will present in the following.

- Now, we additionally consider a set of swaps with floating payments linked to EURIBOR rates with a given tenor. Note that we do not assume that it is possible to invest at this rate, especially not riskless! One might include an asset replicating an investment at this rate later on to explain the results or to model the dynamics of such a system, but then one has to take account of the related default risk as well.

- Those swaps are contracts exchanging fixed payments versus floating payments linked to an exogenous index, the payments themselves are not exposed to credit risk as one usually considers collateralized swap contracts. In particular, they are not exposed to the credit risk causing the basis spread. Furthermore, based on the previously mentioned literature discount factors from OIS discounting have to be used when prizing those collateralized contracts.

- The value of fixed riskless payments is given based on the bootstrapping procedure in the previous section. Thus, what we can extract from the set of swaps considered here is the implied (expected) value of the floating payments.

- Stated differently, in this part of the bootstrap we are not concerned with the "value of a fixed payment in the future" anymore. Now, we try to extract a "description" of the additional swap curve/the related EURIBOR rate which allows us to value arbitrary products linked to those floating payments accordingly. In other words, we have to extract the "value of a floating payment linked to that rate", i.e. the value of the floating payments paid in those contracts. As curves are often described using a discount curve, we will also describe this curve in terms of discount factors later on, even though...
we are merely interested in the forward rates determining the floating payments.

Let \( L^{3M}(T_i, T_{i+1}) \) denote the EURIBOR rate for the time period between \( T_i \) and \( T_{i+1} \) (in our example, the tenor is assumed to be 3 month), which is fixed at \( T_i \) and payed at \( T_{i+1} \) in a swap. As in Section 2.2, the idea would be to iteratively construct portfolios using the given instruments, which exchange a fixed payment today, \( PV(0) \), against a payment of \( L^{3M}(T_i, T_{i+1})(T_{i+1} - T_i) \) at \( T_{i+1} \). Using the language of financial mathematics, we can restate

\[
PV(0) = \text{"Today’s value of a future payment of}
L^{3M}(T_i, T_{i+1})(T_{i+1} - T_i)"
\]

\[
= D(0, T_i) E^{Q_{T_{i+1}}} [L^{3M}(T_i, T_{i+1})](T_{i+1} - T_i)
= D(0, T_i) F^{3M}(0, T_i, T_{i+1})(T_{i+1} - T_i),
\] (4)

where \( Q_{T_{i+1}} \) denotes the \( T_{i+1} \)-forward measure using risk-neutral valuation. \( PV(0) \) can also be translated into a related forward rate \( F^{3M}(0, T_i, T_{i+1}) \), i.e. the rate such that a contract exchanging \( F^{3M}(0, T_i, T_{i+1}) \) against \( L^{3M}(T_i, T_{i+1}) \) at \( T_{i+1} \) has value zero today\(^4\).

Those forward rates, respectively the curve of those forward rates \( T_i \mapsto F^{3M}(0, T_i, T_{i+1}) \), are the objects we are interested in. When valuing a specific (collateralized) derivative consistent with market prices, we can replace the unknown floating rate in the calculation by the respective forward rate (see Equation (4)). As before, we end up with the following set of equations the forwards should satisfy to match market prices:

\[
s_i^{3M} \sum_{j=1}^{n_i}(T_j - T_{j-1}) D(0, T_j)
= \sum_{j=1}^{n_i} D(0, T_j) F^{3M}(0, T_{j-1}, T_j)(T_j - T_{j-1}),
\] (5)

for \( 1 \leq i \leq n \), where \( n_i \) corresponds to the number of payment dates\(^5\) for the \( i \)-th swap and \( s_i^{3M} \) the corresponding par spread.

As in the previous section, we might need some interpolation techniques during this bootstrapping procedure, as in every step of the bootstrap mostly either two or four forward values have to be determined. To state the problem in a similar manner as before, where we were concerned with discount factors, it might be favorable to state the problem in terms of a discount factor curve. Therefore, we will encode this information in a “discount factor curve”\(^6\) \( P^{3M}(0, T) \) using the relation

\[
F^{3M}(0, T_{j-1}, T_j) = \frac{1}{T_j - T_{j-1}} \left( \frac{P^{3M}(0, T_j)}{P^{3M}(0, T_{j-1})} - 1 \right).
\]

\(^4\)Consequently, if one wanted to hedge against the uncertainty in future floating rates, one could lock in the forward rate by entering into such a forward contract at no cost.

\(^5\)We again assume floating and fixed leg to have equal payment dates for simplicity. Here, this does not make a difference at all.

\(^6\)Note that this is of course not a real discount curve as the existence of several different discount curves induces arbitrage. Instead, it is just a convenient way to describing curves of interest rates.
This transforms Equation (5) to

\[ s_i^{3M} \sum_{j=1}^{n_i} (T_j - T_{j-1}) D(0, T_j) \]
\[ = \sum_{j=1}^{n_i} D(0, T_j) \left( \frac{P^{3M}(0, T_{j-1})}{P^{3M}(0, T_j)} - 1 \right), \quad 1 \leq i \leq n. \]

Note that here, as the two discount curves are different, the right hand side of the equation does not reduce to a telescoping sum. Replacing \( D \) by \( P^{3M} \) in the previous equation yields the formula commonly used before the crisis in a one curve framework, where the discounting curve also corresponds directly to the considered EURIBOR rate. Obviously, both approaches differ significantly. What is surprising on the first glance is that the resulting forward rates usually do not. However, there is an easy explanation for this. In both cases, the following equations have to be fulfilled:

\[ s_i^{3M} = \sum_{j=1}^{n_i} w^j_i F^{3M}(0, T_{j-1}, T_j), \quad 1 \leq i \leq n, \]

where \( w^j_i \) are weights summing to one, either

\[ w^j_i := \frac{D(0, T_j) (T_j - T_{j-1})}{\sum_{l=1}^{n_i} (T_l - T_{l-1}) D(0, T_l)} \] "one-curve",

or

\[ w^j_i := \frac{P^{3M}(0, T_j) (T_j - T_{j-1})}{\sum_{l=1}^{n_i} (T_l - T_{l-1}) P^{3M}(0, T_l)} \] "dual-curve".

As long as the form of the considered discount curves is similar, the weights should be very similar\(^7\). Thus, the resulting forwards can not differ too much as the weighted sums with very similar weights have to yield the original input swap rates again. For that reason, one might employ the techniques common before the crisis to get a good approximation of the forward rates. However, when valuing swaps, especially forward starting swaps or on the run swaps, the right discount factors, i.e. the discount factors from Section 2, have to be used.

To demonstrate the procedure, the presented techniques were implemented using market data of February 11, 2013. Regarding the interpolation, we employed a raw interpolation scheme, which is linear on the logarithm of discount factors and corresponds to piecewise constant forward rates (see Hagan, West (2006)). Further details are omitted as they do not contribute to a better understanding of the principles presented here. Figure 1 illustrates the EONIA interest rate curve in terms of the corresponding zero rates \( z(0, T) := -\log(D(0, T))/T \). This is the result of the techniques presented in the second section. Additionally, a 3M EURIBOR curve is depicted which results from "wrongly" applying the techniques of Section 2 to a set of swaps.

\(^7\)For example in the fictitious case that one discount curve equals the other discount curve multiplied by a constant, the weights would be equal.
referencing the 3M EURIBOR rates directly, as it used to be common. Indeed, the second curve is uniformly higher because of the higher default risk associated with the longer tenor. Wrongly using the corresponding discount factors consequently would result in wrong prices for most interest rate products. Furthermore, it is obvious that the structure of the involved discount curves is similar, which according to our previous observation results in very similar forward rates for both approaches. The related forward rates for both approaches can be found in Figure 2. To recognize any deviation, the difference between the forward rates is shown in Figure 3. Similar figures for different curves can be found in Bloomberg starting from the <ICVS> screen.

Fig. 1: The EONIA curve computed with the techniques of the second section based on market data of February 04, 2013. Additionally, the 3M EURIBOR curve is plotted.

4 Including foreign currencies

In Section 2, we computed the discount factors corresponding to a specific reference rate, assuming we were able to borrow/invest at that rate. This means we now know today’s value of a fixed payment in the future in our currency, i.e. the currency of the reference rate. The aim of this section is to translate those discount factors for our local currency into consistent discount factors for a foreign currency, which without loss of generality will be denoted by $\mathcal{S}$ and $D^{\mathcal{S}}(t, T)$. In other words, we are interested in the discount factors for a different currency which result from our local discount factors by borrowing/investing in the foreign currency through a set of different products such as FX forwards or cross currency swaps. This is also the curve used when valuing payments in a foreign currency in a contract collateralized in the local currency.

Of course one could directly start from a comparable (in our case overnight) rate in the foreign currency and proceed as in Section 2. However, this need not necessarily be consistent to the discount factors in the currency one starts from (at least not anymore, see e.g. the significantly negative basis spreads for some
Fig. 2: The forward rates for the 3M EURIBOR computed with the techniques of Section 3 is shown and compared with the approximation based on a straight-forward application of older techniques.

Fig. 3: The difference between the two forward rate curves in bp.
cross-currency swaps). Hence, we start from payments in the local currency and "generate" payments in the foreign currency by swapping payments using FX forwards or cross-currency basis swaps (CCSs). If there existed FX forwards for the whole range of considered maturities, we would be done at this point as we could easily solve for the required discount factors using the relation

\[ D^S(0, T) = \frac{D(0, T)}{FX_T} FX_0, \]

where \( FX_0 \) denotes today’s value of one unit of the local currency in the foreign currency and \( FX_T \) the comparable amount for the future date \( T \) determined by the corresponding FX forward. The intuitive meaning of this equation is that the value of 1 in the foreign currency at \( T \) in the local currency is \( 1/FX_T \). Today, this has the value \( D(0, T)/FX_T \) in the local currency which corresponds to \( FX_0 D(0, T)/FX_T \) in the foreign currency.

Unfortunately, FX forwards are not liquidly traded for the whole range of required maturities. For that reason, we have to use cross-currency swaps which makes the procedure considerably more difficult. We thus also need the procedure described in Section 3 since the usually considered (most liquid) CCSs exchange floating payments as EURIBOR in the local currency versus floating payments in the foreign currency. For a more detailed, academic description of the procedure and the related problems the interested reader is referred to Fujii et al. (2010b)\(^8\).

One has to consider the following structure. By entering into a CCS, one exchanges floating payments plus a basis spread in the local currency versus floating payments in the foreign currency. The value of floating payments in the local currency is known from Section 3. However, both, the expected value of the floating payments in the foreign currency, and the respective discount factors, are unknown. Consequently, one can not solve directly for the discount factors.

Instead, one has to enter a second swap, this time a standard interest rate swap in the foreign currency exchanging exactly the floating payments in the foreign currency versus fixed payments in the foreign currency. This "removes" the floating \( \$ \) payments from the structure. The complete payment structure is illustrated\(^9\) in Figure 4. Note that CCSs also involve an initial exchange of the nominal in the two related currencies and the initial amounts are payed back at maturity. This offers a way to "fund" in the foreign

\(^{8}\)Actually, the approach presented here slightly differs from the approach used in Fujii et al. (2010b). Their idea is the following. As we have seen before, the resulting forward rates mainly depend on the form of the discount factor curve. Thus, they compute the forward rates in the foreign currency by simply applying the procedures presented in Sections 2 and 3, starting from a USD overnight reference rate. They then replace the unknown forward rates in our setting by those forward rates. Based on those forward rates, they solve the resulting system of equations for the unknown discount factors. However, it suffers from the same small mistake as our approach does, see Remark 4.1.

\(^{9}\)Note that for some currency combinations, the structure might be even more complex. Since it is desirable to use the most liquid contracts, one might not be able to swap the floating foreign payments directly into fixed payments. Instead, one swaps them into floating payments of a different tenor, which then can be swapped into fixed payments.
currency. As for example demand to fund in US-\$ is higher than supply, this resulted in the considerably negative basis spreads for USD/EUR CCS. The result of the described bootstrapping procedure basically yields the interest rate one has to pay on \$-amounts borrowed by entering into a CCS.

![Diagram](image.png)

Fig. 4: Illustration of payment streams in the considered combination of a CCS and a \$-IRS.

Under some simplifying assumptions (assuming coinciding payment dates and daycount conventions for all involved swaps, etc.) we end up with the following system of equations that has to be fulfilled in order to match market quotes:

\[
D(0, T_{n_i}) + \sum_{j=1}^{n_i} D(0, T_j) \left( F(0, T_{j-1}, T_j) + bs_i \right) (T_j - T_{j-1}) = D^\$\left(0, T_{n_i}\right) + c_i \sum_{j=1}^{n_i} D^\$(0, T_j) (T_j - T_{j-1}),
\]

for \(1 \leq i \leq n\), where \(c_i\) denotes the par spread of the \$-IRS and \(bs_i\) the basis spread of the involved CCS, which is payed on top of the floating payments in the local currency (depending on the convention of the CCS). The left hand side of these equations represents the value of all payments received by party A and the right hand side the value of all payments made by party A (excluding the value of the initial payments offsetting each other). Note that the floating \$ payments do not show up in these equations. When iteratively solving the system of equations one will have to employ the previously mentioned bootstrapping procedures again.

In Figure 5, the result for the previously used exemplary day is shown. We present the USD curve consistent with the EONIA curve which is derived using CCSs and FX forwards. For comparison, a directly computed USD curve is added. It is obvious that it makes a huge difference which of the following curves is
used. Interestingly, the resulting curve resembles more a shifted USD OIS curve than the EONIA curve we start from, though no USD OIS market quotes are used.

Fig. 5: The EONIA curve plus the consistent corresponding USD curve derived by the presented procedure using CCSs and FX forwards. The USD OIS curve which is computed by applying the procedure presented in Section 2 directly to swap data in the foreign currency is added for comparison.

Remark 4.1
Actually, there occurs a small mistake in this computation. However, it seems to be unavoidable. By setting those payments against each other and in particular, by using the results from Section 2 and 3, we implicitly assume that the collateral underlying the CCS and the foreign currency interest rate swap is also posted in EUR, the local currency. However, this is usually not the case and the market quotes available and used in the bootstrap normally correspond to USD-collateralized contracts.

References


\[ t = T_0 \quad t = T_1 \quad t = T_2 \quad \ldots \quad t = T_{n-1} \quad t = T_n \]

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<td>( \ddots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>Replication of ( s_n(T_{n-1} - T_{n-2}) )</td>
<td>( D(0, T_{n-1}) \cdot D(0, T_{n-2}) )</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>(-s_n(T_{n-1} - T_{n-2}))</td>
</tr>
<tr>
<td>Swap with maturity ( T_n )</td>
<td>0</td>
<td>( 1 - B(T_1)/B(T_0) ) + ( s_n(T_1 - T_0) )</td>
<td>( 1 - B(T_2)/B(T_1) ) + ( s_n(T_2 - T_1) )</td>
<td>( \ldots )</td>
<td>( 1 - B(T_{n-1})/B(T_{n-2}) ) + ( s_n(T_{n-1} - T_{n-2}) )</td>
</tr>
<tr>
<td>Bank account</td>
<td>(-B(T_0))</td>
<td>( B(T_1)/B(T_0) - 1 )</td>
<td>( B(T_2)/B(T_1) - 1 )</td>
<td>( \ldots )</td>
<td>( B(T_{n-1})/B(T_{n-2}) - 1 ) + ( 1 )</td>
</tr>
<tr>
<td>Portfolio</td>
<td>( s_n \sum_{j=1}^{n-1} (T_j - T_{j-1}) D(0, T_j) )</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>( 1 + s_n(T_n - T_{n-1}) )</td>
</tr>
</tbody>
</table>

Table 2: Payments of the considered portfolio at every relevant date.