Abstract

Assuming the absence of arbitrage in a single-name credit risk model, it is shown how to replicate the risk-free bank account until a credit event by a static portfolio of a bond and infinitely many credit default swaps (CDS). From the viewpoint of classical arbitrage pricing theory this static portfolio can be viewed as the solution of a credit risk hedging problem whose dual problem is to price the bond consistently with the CDS. This duality is maintained when the risk-free rate is shifted parallelly. In practice, there is a unique parallel shift $x^* \in \mathbb{R}$ that is consistent with observed market prices for bond and credit default swaps. The resulting, risk-free trading strategy in case of positive $x^*$ earns more than the risk-free rate, is referred to as negative basis arbitrage in the market, and $x^*$ defined in this way is a scientifically well-justified definition for what the market calls negative basis. In economic terms, $x^*$ is a premium for taking the unmodeled residual risks of a bond investment after interest rate risk and credit risk are hedged away, predominantly these are liquidity risk and legal risk.

1 Introduction

Credit default swaps have been introduced to the financial markets in the 1990s as a reaction to a savings and loan crisis in the United States. Since then the market for CDS has experienced significant growth until the Lehman crisis in 2008, when it has suffered a severe setback. Nevertheless, credit default swaps are still traded liquidly and possess a similarly commanding role in the fixed income market as do vanilla put and call options in the equity market. A credit default swap (CDS) is a derivative contract that allows investors to protect against losses resulting from fixed-income investments. Technically, it is an insurance contract between two parties, insurance seller and insurance buyer, referring to a third party, the reference entity. The insurance buyer pays an annualized insurance premium to the insurance seller, who in return is committed to make a compensation payment in case a certain contractually defined credit event takes place with respect to the reference entity. The most typical credit event is a failure-to-pay event, i.e. the reference entity fails to make a due payment on one of its outstanding bonds or loans. In such case, a CDS auction\(^1\) is held, in which all outstanding CDS contracts on the respective reference entity are settled. In this standardized auction process, insurance sellers and buyers determine a recovery rate $R \in [0, 1]$, which intuitively represents the residual value of a bond issued by the reference entity (given as percentage of its nominal). Consequently, each insurance seller has to pay the cash amount $1 - R$ per unit CDS nominal to the insurance buyer.

The motivations for entering credit default swaps differ signifi-
cantly among market participants. Most obviously, (large) fixed income investors are natural insurance buyers, while investors seeking to earn credit risk premia are natural insurance sellers. However, CDS are also used by sophisticated investors as an instrument to set up more technically oriented trades, which do not primarily focus on the default risk, as explained in the sequel. While the payoffs of standard puts and calls refer directly to the value of their underlying equity, a CDS contract is a standardized contract referring to an issuing company, and not directly to a specific bond that one might seek credit protection for. This implies certain imperfections when credit default swaps are used to hedge a bond investment. On the one hand, it is well possible that a reference entity fails to make a due payment only on one bond but serves its others, in which case a CDS protection buyer receives a payment although he might not necessarily have experienced a loss. On the other hand, bond prospectuses are less standardized than the CDS documentation, which might lead to certain “legal gaps” in the CDS insurance. Typical examples are certain collective action clauses which allow a majority of bond holders to change the bond documentation to the disadvantage of a minority. This can lead to losses on the bond that do not trigger a CDS credit event and are thus not compensated for by CDS protection. It is very difficult to accurately take into account all the diverse causes of this imperfectness between bond and CDS market in a pricing model, and standard pricing models simply ignore them. One noticeable exception is Brigo et al. (2014), who discuss the particular effect of collateralization and counterparty default risk on the pricing of CDS, which is one particular aspect (among several) explaining a particular share of the bond-CDS discrepancy. In the present article, we do not seek to explain observed price discrepancies between bond and CDS by a non-standard methodology, because, honestly speaking, due the the numerous risk factors of different nature this is a herculean task. Instead, our goal is rather to use easy-to-implement, and thus practically useful, standard pricing methods to quantify the bond-CDS price discrepancy accurately, because already this mere quantification is a non-trivial task. Morini, Prampolini (2011) point out that such measurement plays a dominant role when pricing derivatives in the presence of funding costs and counterparty credit risk.

Some market participants actively trade the price discrepancies between bonds and CDS. Loosely speaking, if the CDS is cheap relative to an associated bond, one speaks of a negative (bond-CDS) basis, and if it is expensive relative to a bond, one speaks of a positive (bond-CDS) basis. Traders in this market quantify the aforementioned relative price discrepancy by measurements that lack scientific foundation and must be seen as rule-of-thumb formulas that can be quite misleading in particular situations. The most prominent quantification is the difference between bond Z-spread and CDS par spread, which is even implemented on the Bloomberg screen YAS, see also Choudhry (2007). The practically-oriented article Bernhart, Mai (2016) points out the deficiencies of common measurements and proposes a decent alternative, which, however, is only justified heuristically.
The contribution of the present article is to provide the scientific backbone to this measurement. In order to achieve this, the strategy is to consider a slightly idealized model setting, in which infinitely many assets are available, namely credit default swaps with arbitrary maturity. Although this might appear unrealistic from a practical point of view, we demonstrate that this assumption is quite natural from a theoretical viewpoint, since in a standard credit risk pricing model it is just what’s needed to hedge away default risk completely, thus isolating the residual (unmodeled) risks causing the aforementioned price discrepancies, and thereby paving the way to measuring them accurately.

The remainder of the article is organized as follows. Section 2 introduces the model and the associated notations. Section 3 derives a static pricing-hedging duality within the model. Section 4 uses the model setup to rigorously define and introduce what the market calls “negative basis arbitrage”. Section 5 concludes, and an appendix contains the technical proofs.

2 The model

We consider a stochastic model for a financial market on which credit default swaps with all maturities \( t \in [0, T] \) and an eligible bond with maturity \( T \) are traded. To this end, we introduce the following notation:

- \( c \geq 0 \) denotes the annualized coupon rate of the bond, paid continuously,
- \( s \geq 0 \) denotes the annualized coupon rate of each \(^2\) CDS, paid continuously,
- \( \tau \), a random variable formally defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), denotes the random future time point at which the bond defaults (provided \( \tau \leq T \)) and the CDS with maturities \( t \geq \tau \) trigger, and we assume that \( \mathbb{P}(\tau > 0) = 1 \),
- \( R \in [0, 1) \) denotes the (non-random) recovery rate of the bond and each CDS at \( \tau \) (provided \( \tau \leq T \)),
- \( r(t) \) denotes a deterministic short rate assumption, used for discounting future cash flows,
- \( b \geq 0 \) denotes the price of the bond, quoted in percent of its nominal,
- \( u_t \) denotes the price (also known as \textit{CDS upfront}) of the CDS with maturity \( t \in [0, T] \), quoted in percent of its nominal\(^3\).

We collect a few remarks regarding the more crucial and less crucial assumptions of the model:

- **Continuous coupon payments:** In the real market, bond and CDS coupons are paid discretely rather than continuously. The assumption of continuous payments, however,

\(^2\)This rate is standardized in the market, typically \( s = 1\% \) or \( s = 5\% \), and it is always identical across all available maturities for one given reference entity.

\(^3\)A CDS with maturity \( t = 0 \) is a “non-asset”, since \( \mathbb{P}(\tau > 0) = 1 \), but it is formally included by consistently assuming \( u_0 = 0 \).
helps us to derive a nice formula for the optimal hedging portfolio in the sequel. The analogous formula in the more realistic case of discrete payments can be seen as discrete approximation of the formula in the idealized, continuous case. Hence, our idealized formula is still useful, in particular when studying its qualitative nature.

- **Bond covenants:** The bond is assumed to be bullet, i.e. issuer and holder have no early redemption rights and the holder has no conversion-into-equity right. The inclusion of such implicit derivatives destroys the static nature of the model, on which the main results of this article rely. In particular, since credit default swaps are standardized to have no early redemption feature, early redemption covenants of the bond naturally increase the economic discrepancy between bond and CDS, and thus make the hedging problem considered below significantly more difficult. However, this might be an interesting aspect for further research, especially since many newly issued bonds come equipped with such covenants especially in the European market, because the long-lasting low interest rate environment allows issuers to place more and more issues with investors despite of such issuer-friendly covenants.

- **Deterministic interest rate:** We assume that interest rate risk resulting from an investment in the bond is hedged away via interest rate derivatives. Concretely, we assume that the nominal amount of each single cash flow from the bond is perfectly interest rate-hedged via an interest rate forward contract. Since we assume continuous coupon payments, this means that there are infinitely many interest rate forwards in place. Since the perfect interest rate hedge can be approximated fairly well via interest rate swaps and forwards in practice, this idealized assumption is not too severe from a practical perspective. From a theoretical perspective, this assumption is necessary, since we seek to measure remaining risks after interest rate risk is (completely) eliminated. The deterministic short rate $r(t)$, bootstrapped from observed interest rate swap/forward data, must then be viewed as the interest rate that is “locked in” with this perfect interest rate hedge.

- **Recovery rate:** In reality, the recovery rate needs not be equal for bond and CDS, and furthermore is random in general. However, when hedging a bond investment with a CDS, it is always possible to opt for physical delivery in the CDS auction after a credit event. In this case, one receives perfectly matched compensation for the nominal loss on the bond from the CDS. In other words, the CDS protection buyer has the right to opt for equality of bond and CDS recovery. In fact, the CDS implicitly has the so-called cheapest-to-deliver option⁴, i.e. the bond recovery may even be higher than the

⁴A protection buyer can sell her bonds in the market place and instead buy the cheapest available bond prior to the auction. She then delivers the cheaper bond into the auction and the price difference between the two bonds is thus consumed as a gain.
CDS recovery, but we neglect this optionality throughout.

- **CDS maturities:** In reality, CDS are available for a finite number of maturities, say $0 < t_1 < t_2 < \ldots < t_m$. The idealized assumption of CDS with arbitrary maturity is somehow algebraically consistent with the infinitely many possible outcomes for $\tau$, which might take values in all of $(0, \infty]$. From the viewpoint of arbitrage pricing theory, it may on first glimpse appear problematic to assume the existence of infinitely many assets. However, the opposite is true for our purpose. Intuitively, it corresponds to a completion of the market model, since the flexibility to trade in CDS with arbitrary $t$ gives enough flexibility so that the credit risk resulting from the timing of the random variable $\tau$ can be eliminated, as we will see. In order to apply our result to real data, observed upfront prices $u_{t_1}, \ldots, u_{t_m}$ need to be interpolated to retrieve a full continuum $u_t$, $t \in [0, T]$, of upfront prices with $u_0 = 0$. Typical ways of accomplishing such interpolation is a simple linear interpolation of the observed prices, or interpolation by assuming certain parameterizing functions of the distribution function of $\tau$ to be piecewise constant, see Hull, White (2000); O’Kane, Turnbull (2003). We assume that the function $t \mapsto u_t$ is right-continuous and of finite variation, which is clearly a mild, non-severe assumption, satisfied by the aforementioned interpolation methods (which even yield a continuous function $t \mapsto u_t$).

**Remark 2.1 (Discontinuities in $t \mapsto u_t$)**

For the practical applications of our presented results which we have in mind a continuous interpolation of the observed CDS prices is sufficient. Nevertheless, we find it educational to point out that we explicitly allow the possibility for discontinuities to avoid technical and unnecessary, thus possibly confusing, restrictions. A potential jump in the given upfront prices at $t \in (0, T]$ intuitively corresponds to the market believing that there is a positive probability that $\tau$ happens precisely at $t$. In economic terms such atoms are natural, because a quite typical time point when a failure-to-pay credit event takes place is the end of a grace period of a due coupon payment, which is well-known in advance by market participants. Nevertheless, such atoms are typically ruled out in standard credit risk pricing models, in particular in intensity-based models, as a matter of convenience, a noticeable exception being Gehmlich, Schmidt (2018).

Given the available instruments in our market model, a static portfolio is a triplet $(\varphi_b, \varphi_c, \varphi)$ of a constant $\varphi_b$ that specifies the bond nominal, a constant $\varphi_c$ that specifies the nominal of the CDS with maturity $T$, and a right-continuous finite-variation function $\varphi: [0, T] \to \mathbb{R}$ with the interpretation that $d\varphi(t)$ specifies the nominal of the CDS with maturity $t \in [0, T)$. Throughout, we always normalize the portfolio in such a way that $\varphi_b = 1$, since our focus is to use the available CDS contracts for minimizing the risk exposure resulting from a (long) bond investment. More abstractly speaking, we view the credit default swaps as primary assets and seek to hedge the exposure resulting from a bond investment, which is viewed as a secondary asset. Consequently,
we identify a portfolio \((1, \varphi_c, \varphi)\) as a pair \((\varphi_c, \varphi)\) in the sequel, always implicitly meaning that \(\varphi_c = 1\). The value \(V_0^{\varphi_c, \varphi}\) of the portfolio \((\varphi_c, \varphi)\) at inception \(t = 0\) is given by

\[
V_0^{\varphi_c, \varphi} := b + \varphi_c u_T + \int_0^T u_t d\varphi(t).
\]

With a sequence \(\{t_k^{(n)}\}_{k=0,\ldots,n}, n \in \mathbb{N}\), of refining partitions \(0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T\) of the interval \([0, T]\), we point out that the Riemann-Stieltjes definition of the integral in (1), which only determines the holdings in credit default swaps with maturities \(t \in [0, T]\), which is why the holding in the CDS with maturity \(T\) is declared separately by \(\varphi_c\). In practice, the function \(\varphi\) needs to be approximated by a step function, which is identically constant on those intervals \(t \in (a, b)\) for which no CDS are available, and which then turns the integral into a finite sum. The finite variation and right-continuity assumptions on \(u_t\) and on \(\varphi\) justify the existence of the integral. In economic terms, finite variation of \(\varphi\) means that the total invested amount \(V_0^{\varphi_c, \varphi}\) is finite. Below in Lemma 3.1 and Corollary 4.2, we are particularly interested in differentiable functions \(\varphi\) but, as already mentioned, in practice the function \(\varphi\) is naturally a step function.

Analogously, we denote the random net present value of all cash flows from the portfolio \((\varphi_c, \varphi)\) by

\[
V^{\varphi_c, \varphi}(\tau) := B(\tau) + \varphi_c U_T(\tau) + \int_0^T U_t(\tau) d\varphi(t),
\]

where the random net present values of all cash flows from bond and CDS are given by

\[
B(\tau) := 1_{\{\tau > T\}} e^{-\int_0^T r(t) dt} + 1_{\{\tau \leq T\}} R e^{-\int_0^\tau r(t) dt} + c \int_0^{\min\{T, \tau\}} e^{-\int_0^y r(t) dt} dy,
\]

\[
U_t(\tau) := 1_{\{\tau \leq t\}} (1 - R) e^{-\int_0^\tau r(t) dt} - s \int_0^{\min\{t, \tau\}} e^{-\int_0^y r(t) dt} dy, \quad t \in [0, T].
\]

Notice in particular that \(t \mapsto U_t(\tau)\) is right-continuous of finite variation almost surely, and the usual Riemann-Stieltjes definition applies to (2) in the same way as for (1). The discounted profit and loss of the portfolio on \([0, T]\) is finally obtained when subtracting the initial cost, that is by \(V^{\varphi_c, \varphi}(\tau) - V_0^{\varphi_c, \varphi}\).

From a stochastic modeling point of view, the sole source of randomness in our model is the default time \(\tau\). For the purpose of the present article there is no need to further specify the probability space on which \(\tau\) is defined. In particular, we only consider static portfolios \((\varphi_c, \varphi)\) as described, so for our purpose we do not need an information flow model to describe possibly changing market opinions on \(\tau\) over time (possibly resulting in dynamic
rebalancings of the portfolio). From this perspective, our purely static considerations are model-free. The crucial (implicit) assumption within our setting is that the cash flows of bond and CDS are known functions of \( \tau \), which is a simplification of reality in the sense that it (i) rules out uncertainty about the aforementioned “legal gaps”, and (ii) ignores liquidity/funding/collateral aspects, such as partially considered in Brigo et al. (2014) for instance. Our goal is to accurately measure precisely those (unmodeled) risks of a bond investment that are still present after the risk resulting from \( \tau \) is hedged away completely using credit default swaps. A quantification for these risks is what the market calls bond-CDS basis, and we address this problem in Section 4 below.

3 Pricing-hedging duality

The most typical and most obvious CDS hedge in the marketplace specifies \( \phi_c = y \) for some value \( y \) near one, and \( \phi \equiv 0 \). In words, a bond investment is complemented by a maturity-matched CDS whose nominal \( y \) is chosen either equal to one or close to one, depending on the market prices \( b \) and \( u_T \) as well as the portfolio manager’s preference. While the nominal-matched choice \( y = 1 \) corresponds to a hedge that provides perfect default insurance close to maturity, other choices might be preferred if the “package price” \( b + u_T \) for bond and maturity-matched CDS trades significantly away from par. The resulting net present value of the portfolio \((y,0)\) in dependence on the default timing is then given by

\[
V^{y,0}(\tau) = (c - ys) \int_0^{\min(T,\tau)} e^{-\int_0^y r(t) dt} dy + \left( 1_{\{\tau \leq T\}} (R + y(1 - R)) + 1_{\{\tau > T\}} \right) e^{-\int_0^{\min(T,\tau)} r(t) dt}.
\]

In particular, there is a discontinuity at \( \tau = T \) in case \( y \neq 1 \). Depending on \( c, s, r(t), y \), and on the market prices \( b \) and \( u_T \), this shows that the timing of a potential credit event makes a crucial difference for the resulting profit and loss, see also Figure 1 below. In other words, this common CDS hedging strategy does not hedge away credit risk completely in general. In contrast, Lemma 3.1 below and its corollary show how this can be achieved. To this end, we introduce the following assumption:

(A) We assume that the observed CDS prices are arbitrage-free.

This means there exists an equivalent probability measure \( Q \sim P \) such that

\[
u_t = \mathbb{E}^Q[U_t(\tau)], \quad t \in [0,T].
\]

From a theoretical viewpoint, in the following lemma we assume that the credit default swaps are the primary assets and Assumption (A) means that the market for these primary assets is arbitrage-free. On the contrary, the bond is viewed as a secondary asset that we seek to hedge with the primary assets. We demonstrate that this is possible if and only if the observed bond price \( b \) is determined by the observed CDS prices, in which case the bond can be replicated by a portfolio of CDS. In other words, the problem of hedging the default exposure from the bond with
CDS is somehow dual to the problem of pricing the bond consistently, which is further explained in Remark 3.4 below.

**Lemma 3.1 (Duality of pricing and hedging)**
We assume that (A) is valid. The following statements are equivalent.

(a) There is a differentiable function \( \varphi : (0, T) \to \mathbb{R} \) such that the discounted profit and loss of the static portfolio \((1, \varphi)\) is identically zero, that is \( V^{1,\varphi}(\tau) \equiv V^{1,\varphi}_0 \).

(b) The bond price \( b \) is consistent with the observed CDS prices, that is \( b = \mathbb{E}^Q[B(\tau)] \).

**Proof**

See the Appendix.

We define for \( t \in [0, T] \) the function

\[
F_0(t) := 1 - e^{-\frac{s}{1-R}t} + \frac{e^{-\frac{s}{1-R}t}}{1-R} \int_t^T e^{\int_u^T \frac{s}{1-R} + r(v) dv} d\mu_u.
\]

It is not difficult to observe that \( F_0 \) satisfies

\[
d\mu_t = e^{-\int_0^t r(u) du} ((1-R) dF_0(t) - s \{ 1 - F_0(t) \} dt),
\]

meaning that \( F_0(t) = \mathbb{Q}(\tau \leq t) \) for \( t \in [0, T] \) under Assumption (A). From this observation and from the proof of Lemma 3.1 we can extract the following interesting corollary.

**Corollary 3.2 (The optimal static CDS hedge)**

Assume (A) and that one (hence both) of the statements in Lemma 3.1 are true. The optimal function \( \varphi \) referred to in statement (a) may be chosen as

\[
\varphi(t) := \int_t^T e^{-\int_u^T \frac{s}{1-R} + r(v) dv} \frac{s + r(u) - c}{1-R} du,
\]

for \( t \in (0, T) \).

**Proof**

The expression for \( \varphi \) is an immediate by-product of the proof of Lemma 3.1.

From the second fundamental theorem of asset pricing in classical arbitrage pricing theory we know that uniqueness of a risk-neutral pricing measure is essentially equivalent to market completeness. By our assumption of the existence of infinitely many CDS, the distribution function \( F_0 \), and thus \( \mathbb{Q} \), is essentially unique. Statement (a) in Lemma 3.1 can be re-phrased to state that the bond, viewed as a secondary asset, is attainable by a static portfolio of primary assets, i.e. can be replicated by a portfolio of credit default swaps and the risk-free bank account. In other words, by assuming the existence of a continuum of CDS contracts, we have completed the market model and put ourselves in the position to hedge away the risk exposure resulting from a default of the bond investment perfectly.
Remark 3.3 (The credit-triangle)
In the special case when the observed CDS prices satisfy \( u_t \equiv 0 \), i.e. the upfronts for all maturities are identically zero, the CDS coupon \( s \) is equal to the so-called CDS par spread. It is observed that \( F_0 \) equals the distribution function of an exponentially distributed random variable with intensity \( s/(1-R) \). With a first-order Taylor approximation of the exponential function one obtains

\[
F_0(1) = 1 - e^{-\frac{s}{1-R}} \approx \frac{s}{1-R},
\]

which results in the well-known credit-triangle equation

\[
s = F_0(1) (1-R),
\]

which is a memory hook equation, popular with credit market participants, that relates annual default probability \( F_0(1) \), CDS par spread \( s \), and loss given default \( R \).

Remark 3.4 (Duality in the discrete case)
(Bernhart, Mai, 2016, Lemma 1) can be viewed as a discrete version of Lemma 3.1, respectively of Corollary 4.2 below. In order to draw the link to this result, the (possibly continuous) state space of the random variable \( \tau \) needs to be assumed to consist of finitely many values \( t_1 < \ldots < t_d < \infty \) with \( t_d > T \), and we only assume the existence of finitely many CDS contracts with maturities in between these potential values. The equation \( V^1 \varphi(\tau) = V_0^1 \varphi \) then becomes a linear equation system of the form \( A \vec{y} = \vec{0} \) under this discretization, where the unknown variable \( \vec{y} \) corresponds to \( \varphi \) and \( A \) is a square matrix. In a heuristic manner, the proof of (Bernhart, Mai, 2016, Lemma 1) points out that the linear equation system resulting from a discretization of the equations \( b = \mathbb{E}^Q[B(\tau)] \) and \( u_t = \mathbb{E}^Q[U_t(\tau)] \) is of the form \( A \vec{y} = \vec{0} \), with \( \vec{y} \) corresponding to the discretized probability distribution of \( \tau \) under \( Q \) and \( A^t \) is the transpose of \( A \). In words, the two statements (a) and (b) become dual linear equation systems under an appropriately chosen discretization, so that (non-)existence of a solution to one system is equivalent to (non-)existence of a solution to the other.

4 Negative basis arbitrage
In practice, observed CDS prices across different maturities are virtually always arbitrage-free, so that the assumption of Lemma 3.1 is no loss of relevance. However, the observed bond price \( b \) is hardly ever consistent with the CDS prices, so that statement (b), hence (a), in Lemma 3.1 is typically not satisfied. In particular, if CDS prices are cheap relative to the bond price, the strategy of buying the bond and hedging it via too cheap CDS protection is known as the negative basis arbitrage strategy, see Choudhry (2007). There can be two reasons for the observed price discrepancy: either there actually is arbitrage, or the modeling approach of considering \( \tau \) as the sole source of randomness is overly simplistic and misses essential risk components, which are reflected in market prices. Indeed, in Bernhart, Mai (2016) and the references therein one can find good explanations for the existence of the observed price discrepancy. Roughly classified,
aside from market frictions (i.e. arbitrage) the dominant reasons in practice are legal issues (there are certain adverse scenarios, in which CDS protection fails) and liquidity (e.g. the bond investment binds capital so that certain market participants prefer to enter credit-risky positions by selling CDS, making CDS protection cheap relative to the bond). Both legal risks and liquidity risks are non-trivial to take into account in a pricing model.

However, bond and CDS prices can be used to “measure” the aforementioned legal and liquidity risks accurately, as will be pointed out. This measurement has first been introduced in Bernhart, Mai (2016) by a heuristic argument which is made rigorous below. The crucial idea is to stick with the simple model and leave legal and liquidity risks unmodeled, but to introduce a premium for taking these risks into the model. This premium is introduced as a parallel shift of the risk-free rate \( r(t) \) and can be backed out numerically, serving as measurement for legal and liquidity risks associated with the investment.

The optimal strategy in Corollary 3.2 replicates the risk-free bank account by a portfolio of risky assets. In fact, statement (a) in Lemma 3.1 means that the portfolio specified by \((1, \varphi)\) accrues at the risk-free rate \( r(t) \) until \( \min\{\tau, T\} \). If we apply Lemma 3.1 with the risk-free rate \( r(t) + x \) for a positive parallel shift \( x \), the portfolio in statement (a) accrues at the risk-free rate \( r(t) + x \), i.e. is an arbitrage in the real market. An application of Lemma 3.1 implies that such arbitrage exists if we find a probability measure \( Q \sim P \) such that

\[
\begin{align*}
B_x^x(\tau) := 1_{\{\tau > T\}} e^{-\int_0^\tau x + r(t) \, dt} + 1_{\{\tau \leq T\}} R e^{-\int_0^\tau x + r(t) \, dt} \\
&\quad + c \int_0^{\min\{T, \tau\}} e^{-\int_y^\tau x + r(t) \, dt} \, dy,
\end{align*}
\]

\[
U_t^x(\tau) := 1_{\{\tau \leq t\}} (1 - R) e^{-\int_0^\tau x + r(t) \, dt} \\
&\quad - s \int_0^{\min\{t, \tau\}} e^{-\int_y^\tau x + r(v) \, dv} \, dy, \quad t \in [0, T].
\]

Indeed, in practice one typically finds a unique \( x = x_* \in \mathbb{R} \) such that the condition (3) is satisfied, as the following technical lemma shows. If this unique \( x_* \) happens to be positive, one has found the described arbitrage.

**Lemma 4.1 (Existence of \( x_* \))**

We assume that (A) holds true. For each \( x \in \mathbb{R} \) we introduce the function

\[
F_x(t) := 1 - e^{-\frac{x}{1-R} t} + \frac{e^{-\frac{x}{1-R} t}}{1-R} \int_0^t e^{\int_0^y \frac{x}{1-R} + r(v) \, dv} \, dy,
\]

for \( t \in [0, T] \).

(a) For each \( x \in \mathbb{R} \) and \( t \in [0, T] \) we have

\[
\begin{align*}
u_t &= (1 - R) \int_0^t e^{-\int_0^y x + r(v) \, dv} \, dF_x(y) \\
&\quad - s \int_0^t e^{-\int_0^y x + r(v) \, dv} (1 - F_x(y)) \, dy.
\end{align*}
\]
(b) The equation

\[ b = (1 - F_x(T)) e^{-\int_0^T x + r(t) \, dt} + R \int_0^T e^{-\int_0^y x + r(v) \, dv} \, dF_x(y) \]

\[ + c \int_0^t e^{-\int_0^y x + r(v) \, dv} (1 - F_x(y)) \, dy \]

in the variable \( x \in (-1/T, \infty) \) either has no solution, or exactly one solution \( x = x_* \), since the right-hand side of the equation is strictly decreasing in \( x \in (-1/T, \infty) \).

(c) If a solution \( x_* \) in (b) exists and happens to be such that \( F_{x_*} \) is non-decreasing, right-continuous with \( F_{x_*}(T) \leq 1 \), then (3) is satisfied and \( F_{x_*} \) equals the distribution function of \( \tau \) under some equivalent probability measure \( \mathbb{Q} \sim \mathbb{P} \).

Proof
See the Appendix. \( \square \)

The negative basis arbitrage can now be formulated as in the following corollary. To this end, we call a portfolio \((\phi_c, \phi)\) jump-to-default neutral if an immediate default causes no profits and losses, i.e. if \( V^{\phi_c, \phi}(0) = V^{\phi_c, \phi}_0 \). Intuitively, the portfolio is instantaneously PnL-neutral with respect to a credit event.

Corollary 4.2 (Negative basis arbitrage)
Assume that observed market prices are such that \( x_* \) from Lemma 4.1 exists. Then with

\[ \varphi_*(t) := \int_t^T e^{-\int_t^u s + x_* + r(v) \, dv} \frac{s + x_* + r(u) - c}{1 - R} \, du \]

the portfolio \((1, \varphi_*)\) is jump-to-default neutral and the derivative of the function \( \tau \mapsto V^{1, \varphi_*}(\tau) \) on \( \tau \in (0, T) \) is

\[ \left\{ \begin{array}{ll}
 \text{positive} & \text{if and only if } x_* > 0 \\
 \text{zero} & \text{if and only if } x_* = 0 \\
 \text{negative} & \text{if and only if } x_* < 0
\end{array} \right. \]

More precisely,

\[ \frac{\partial}{\partial \tau} V^{1, \varphi_*} = x_* e^{-\int_0^\tau r(u) \, du} \left( 1 - (1 - R) \varphi_*(\tau) \right). \]

Proof
See the Appendix. \( \square \)

In practice, the assumption of Corollary 4.2 is virtually always met and \( F_{x_*} \) is a distribution function, i.e. the situation of Lemma 4.1(c) holds true. Intuitively, the case \( x_* < 0 \) is interpreted as CDS protection being expensive relative to the bond price \( b \). One typical explanation for this observation could be that the cheapest-to-deliver option of the CDS has a non-negligible value and the bond with price \( b \) is not equal to the cheapest-to-deliver bond. The case \( x_* > 0 \) means that CDS protection is cheap relative to the bond price \( b \), and the static portfolio \((1, \varphi_*)\) constitutes an arbitrage in our model. As already explained, sometimes this observation actually indicates an arbitrage opportunity in the
real world, but often it indicates certain insurance gaps in the CDS protection and/or unmodeled liquidity effects. The parallel shift $x_*$ can thus be viewed as a risk premium for taking these legal and liquidity risks when entering the negative basis “arbitrage” trade $(1, \varphi_*)$. Summing up, the following definition is now scientifically justified, and differs from common definitions in the marketplace, such as in Choudhry (2007) or on the Bloomberg screen YAS.

**Definition 4.3 (Negative basis)**

Provided existence, the unique number $x_*$ of Lemma 4.1 is called the negative (bond-CDS) basis associated with the considered bond.

It is important to mention that Definition 4.3 is associated with a bond, not an issuer. Another bond issued by the same entity can have another negative basis. This is clearly a desired property of the measurement, since the risks that are quantified by $x_*$ can be highly issue-specific, and the measurement can be particularly useful to differentiate between several bonds of the same issuer. Furthermore, Definition 4.3 is a relative measurement with respect to the applied risk-free rate $r(t)$. For instance, it makes a crucial difference whether $r(t)$ is bootstrapped from observed interest rate swaps with overnight tenor, or with 3-month-based tenor, both cases being common in practice. Finally, Definition 4.3 is theoretical in the sense that $x_*$ is only unique because of the assumption of infinitely many CDS, which we never have in practice. In reality, $x_*$ depends on the implemented interpolation procedure that extends observed CDS prices to a continuum $t \mapsto u_t$ for all $t \in (0, T]$ (although this dependence is not too critical in practice).

**Remark 4.4 (No re-investment of proceeds)**

In Corollary 4.2 the differential equation for $V^{1,\varphi_*}(\tau)$ could alternatively be written as

$$\frac{\partial}{\partial \tau} V^{1,\varphi_*}(\tau) = x_* \left( V^{1,\varphi_*}(\tau) (c - \{1 - \varphi_*(t)\} s) e^{-\int_0^\tau r(v) dv} dt \right).$$

This shows that instantaneously for very small $\tau$ the function $V^{1,\varphi_*}(\tau)$ behaves like $V^{1,\varphi_*}(0) \exp(x_* \tau)$, i.e. accrues at the continuous rate $x_*$. With increasing $\tau$, the difference between $V^{1,\varphi_*}(\tau)$ and $V^{1,\varphi_*}(0) \exp(x_* \tau)$ increases due to the fact that the potentially received coupons until $\tau$ (from bond and all CDS) are not re-invested into a new negative (or positive) basis position. Instead, these potential proceeds are put into the risk-free bank account and accrue at the rate $r(\cdot)$ (as compared to the rate $r(\cdot) + x_*$, which is earned via $(1, \varphi_*)$).

Let us study the optimal hedge $\varphi_*$ a little further by assuming for the sake of simplicity that $r(t) \equiv r$ is identically constant. We further assume that $x_* \geq 0$ (i.e. there is negative basis) and that $r > -x_* - s/(1 - R)$, which is virtually always satisfied in reality.
In this case, it is readily observed that

\[ \varphi_*(t) = \frac{s + x_s + r - c}{s + (1 - R)(x_s + r)} \left(1 - e^{-\left(\frac{x_s}{\tau - R} + x_s + r\right)(T-t)}\right) \]

\[ \varphi'_*(t) = -\frac{s + x_s + r - c}{1 - R} e^{-\left(\frac{x_s}{\tau - R} + x_s + r\right)(T-t)}. \]

In particular, we observe that

\[ \varphi_*>0 \iff \varphi_*<0 \iff x_*>c-(s+r). \]

In words, this means that if the negative basis \( x_* \) is larger than \( c-(s+r) \), for instance in case of a small bond coupon compared with the CDS coupon, then one sells short-dated CDS in order to enlarge the continuous cash in-flow. Conversely, if the bond coupon is so large that \( c-(s+r) > x_* \), one uses this continuous bond coupon in-flow to buy short-dated CDS in addition to the maturity-matched CDS. In both cases, the absolute value of \( \varphi_* \) increases in \( t \), which means that one sells/buys more CDS the larger \( t \).

We end this article with an illustrative example, whose numbers are inspired by a real-world case. To this end, we assume

\[ r(t) = 0.5\%, \ c = 8.625\%, \ s = 5\%, \ R = 20\%, \ T = 3.85, \]

\[ u_1 = 8.88\%, \ u_2 = 12\%, \ u_3 = 16\%, \ u_4 = 22\%, \ b = 75.26\%, \]

and we interpolate the given CDS prices with maturities 1, 2, 3, 4 linearly, in particular \( u_T = 21.1\% \). This is a negative basis example, as it turns out that \( x_* = 4.5\% \) and \( F_{x_*} \) is a proper distribution function.

Figure 1 illustrates the continuous function \( \tau \rightarrow V^{1,*}(\tau) \), in comparison with the functions \( \tau \rightarrow V^{0,0}((\tau) \) for different values \( y \). Notice that the latter functions for \( y \neq 1 \) have a discontinuity at maturity \( T \). The values \( V^{y,0}(T) \) are depicted as circles in the plot, in order to highlight these discontinuities. Intuitively, these static strategies are by definition not indifferent with respect to default timing. For \( y > 1 \) one hopes that default happens prior to \( T \), while for \( y < 1 \) one hopes that default does not happen prior to \( T \). The observation \( V^{1,*}(T) > V^{1,0}(T) \) intuitively means that in the survival event \( \{\tau > T\} \) the portfolio \((1, \varphi_*)\) has a higher profit than the portfolio \((1, 0)\). Furthermore, it is observed that \((1, \varphi_*)\) is jump-to-default neutral, while \((1, 0)\) and \((1, 1, 0)\) are jump-to-default positive (i.e. profit from an early default), and \((0, 9, 0)\) is jump-to-default negative (i.e. suffers from an early default).

Figure 2 visualizes the function \( \varphi_* \). It is observed that \( \varphi_* \) is decreasing, which means that the static portfolio \((1, \varphi_*)\) sells protection for maturities \( t \in (0, T) \). The sum of bond and nominal and maturity-matched CDS equals \( b + u_T = 96.36\% \). As observed from Figure 1, an immediate default would thus lead to a gain of \( 100\% - 96.36\% = 3.64\% \) for the portfolio \((1, 0)\). In contrast, the portfolio \((1, \varphi_*)\) dispenses with this over-protection and instead uses it to sell more CDS protection, distributed across the maturities \( t \in (0, T) \), thereby trading unnecessary over-protection in for more CDS coupon earnings. In this example, \( V^{1,*}_{1} = 95.7\% \), or, in other words

\[ \int_0^T u_t \, d\varphi_*(t) = 95.7\% - 96.36\% = -0.66\%. \]
5 Conclusion

It was shown how relative price discrepancies between credit default swaps and associated bonds, that result from risk factors other than default timing, could be quantified accurately. The key idea to accomplish this was to complete the market model by considering credit default swaps with arbitrary maturities, which allowed to replicate the bond. As a consequence, remaining price discrepancies after this completion could be quantified uniquely in terms of a parallel shift of the risk-free interest rate used for discounting cash flows. Ultimately, using this methodology it was possible to define what the market calls bond-CDS basis in a mathematically rigorous manner.

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References


F. Gehmlich, T. Schmidt, Dynamic defaultable term structure


**Appendix**

This appendix collects the proofs for the statements made in the main body of the article.

**Proof (of Lemma 3.1)**

We first assume that statement (b) holds true. First observe that the function \( \tau \mapsto V^{1,\varphi}(\tau) \) is continuous on \((0, \infty)\), provided \( \varphi \) is continuous, in particular has no jump at \( T \) and remains identically constant on \([T, \infty)\). Let \( \varphi \) be some finite variation function and observe on the event \( \{ \tau \leq T \} \) that

\[
V^{1,\varphi}(\tau) = \int_0^\tau \left( c - s \left\{ 1 + \varphi(T) - \varphi(t) \right\} \right) e^{-\int_0^t r(u) \, du} \, dt + e^{-\int_0^\tau r(u) \, du} \left( 1 + (1 - R) \left\{ \varphi(T) - \varphi(\tau) \right\} \right).
\]

Now we further specify

\[
\varphi(t) := \int_t^T e^{-\int_t^u \frac{s + r(u) - c}{1 - R} \, du} \, du,
\]

(4)
and observe from this on \( \{ \tau < T \} \) that

\[
\frac{\partial}{\partial \tau} V^{1, \varphi}(\tau) = 0,
\]

so that \( V^{1, \varphi}(\tau) \) is indeed independent of \( \tau \), i.e. a constant. Finally, it is left to verify that \( V^{1, \varphi}(\tau) \) equals \( V^{1, \varphi}_0 \). But this follows precisely from the assumption, since

\[
V^{1, \varphi}_0 = b + u_T + \int_0^T u_t \, d\varphi(t) \\
\overset{(b)}{=} \mathbb{E}^Q[B(\tau)] + \mathbb{E}^Q[U_T(\tau)] + \int_0^T \mathbb{E}^Q[U_t(\tau)] \, d\varphi(t) \\
= \mathbb{E}^Q[V^{1, \varphi}(\tau)] = V^{1, \varphi}(\tau).
\]

Conversely, assume that statement (a) holds true. Re-writing statement (a), we observe that

\[
B(\tau) = V^{1, \varphi}_0 - U_T(\tau) - \int_0^T U_t(\tau) \, d\varphi(t) \\
= b + u_T - U_T(\tau) + \int_0^T u_t - U_t(\tau) \, d\varphi(t).
\]

Taking expectations with respect to \( \mathbb{Q} \) on both sides and using the assumption \( \mathbb{E}^Q[U_t(\tau)] = u_t \) for all \( t \in [0, T] \) then immediately implies the claim. \( \square \)

**Proof (of Lemma 4.1)**

Part (a) is readily verified. We observe from the equation for \( u_T \) that

\[
R \int_0^T e^{-\int_0^T x + r(v) \, dv} \, dF_x(y) \\
= \frac{R}{1 - R} \left( u_T + s \int_0^T (1 - F_x(y)) e^{-\int_0^T x + r(v) \, dv} \, dy \right).
\]

From this we obtain

\[
(1 - F_x(T)) e^{-\int_0^T x + r(t) \, dt} + R \int_0^T e^{-\int_0^T x + r(v) \, dv} \, dF_x(y) \\
+ c \int_0^T e^{-\int_0^T x + r(v) \, dv} (1 - F_x(y)) \, dy = \frac{R}{1 - R} u_T + g_x(T) + \left( c + \frac{s R}{1 - R} \right) \int_0^T g_x(t) \, dt,
\]

with

\[
g_x(t) := e^{-\int_0^t \frac{\tau}{1 - R} + r(v) \, dv} \left( 1 - \frac{1}{1 - R} \int_0^t e^{\int_0^v \frac{\tau}{1 - R} + r(y) \, dy} \, du_v \right)
\]

We show that \( g_x(t) \) is strictly decreasing in \( x \). To this end, we observe that

\[
\frac{\partial}{\partial x} g_x(t) = e^{-\int_0^t \frac{\tau}{1 - R} + r(v) \, dv} \times \\
\times \left( -t + \frac{1}{1 - R} \int_0^t (t - v) e^{\int_0^v \frac{\tau}{1 - R} + r(y) \, dy} \, du_v \right).
\]
so that it is sufficient to prove that the term inside the brackets is smaller than zero. From Assumption (A) we know that the distribution function $F_0$ satisfies the identity

$$\text{d}u_v = e^{-\int_0^v r(y) \text{d}y} \left( (1 - R) \text{d}F_0(v) - s \left( 1 - F_0(v) \right) \text{d}v \right),$$

from which it is not difficult to observe that

$$-t + \frac{1}{1 - R} \int_0^t (t - v) e^{\int_0^v \frac{s}{1 - R} + x + r(y) \text{d}y} \text{d}u_v$$

$$= -\int_0^t \left( 1 - F_0(v) \right) e^{\left( \frac{s}{1 - R} + x \right) v} (1 + x (t - v)) \text{d}v \leq 0,$$

where we made use of the fact that $F_0 \leq 1$ and $x > -1/T$. This finishes the proof of part (b). Part (c) follows, since the equations in parts (a) and (b) are equal to (3), provided $F_{x*}$ equals the distribution function of $\tau$ under $Q$. □

**Proof (of Corollary 4.2)**

The claimed function $\varphi_{x*}$ is exactly the function $\varphi$ from Corollary 3.2, if $r(t)$ is replaced by $r(t) + x_*$. This makes clear from Lemma 3.1 that $V^{1,\varphi_{x*}}(0) = V_0^{1,\varphi_{x*}}$. The computation of the derivative of $V^{1,\varphi_{x*}}(\tau)$ with respect to $\tau$ is straightforward. So left to show is really only that the sign of the derivative equals the sign of $x_*$. To this end, it is sufficient to check that

$$\int_\tau^T e^{-\int_t^\tau \frac{s}{1 - R} + x_* + r(v) \text{d}v} \left( s + x_* + r(t) - c \right) \text{d}t < 1.$$

If the left-hand side is negative, this is trivial. Otherwise, since $s, c \geq 0$ and $R \in [0, 1)$, the expression on the left-hand side can be estimated from above by

$$\int_\tau^T e^{-\int_t^\tau \frac{s}{1 - R} + x_* + r(v) \text{d}v} \left( \frac{s}{1 - R} + x_* + r(t) \right) \text{d}t$$

$$= 1 - e^{-\int_\tau^T \frac{s}{1 - R} + x_* + r(v) \text{d}v},$$

which is smaller than one. □